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GRADE 12 MATHEMATICS TEACHER'S GUIDE

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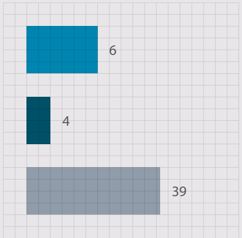

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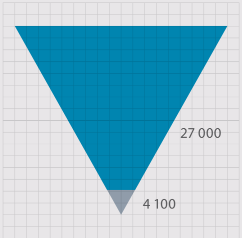
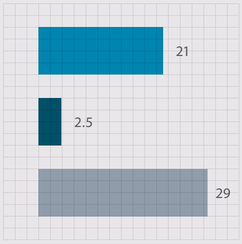


- Litres of ink used in the production of all the grade 10, 11 and 12 textbooks
- Litres of glue used in the production of all the grade 10, 11 and 12 textbooks



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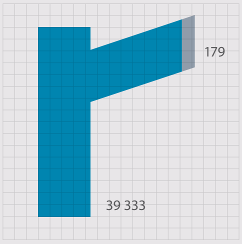

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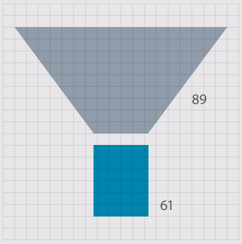
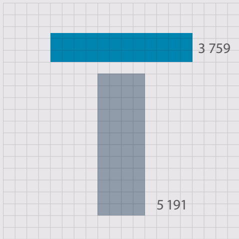
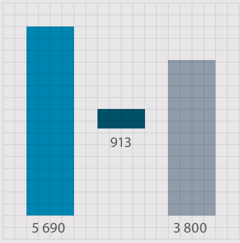


- Hours spent being taught this book
- Hours spent doing homework from this book



- Length of pages side by side (cm)
- Length of pages top to bottom (cm)



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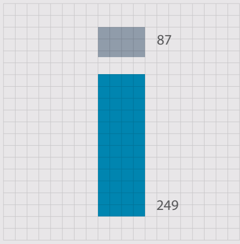
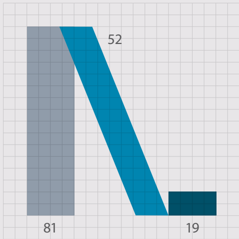
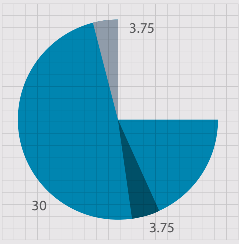
- How many times student picks nose while reading this book
- How many times student clicks pen while reading this book



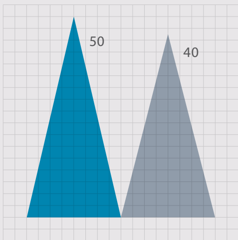
- Number of females who helped write this book
- Number of males who helped write this book



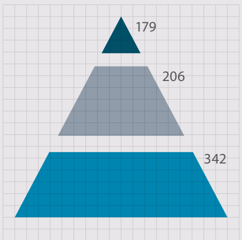
- Masters students who contributed to this book
- Honours students who contributed to this book
- Undergraduate students who contributed to this book



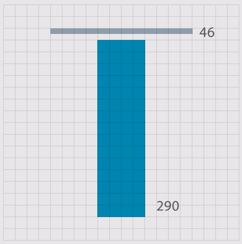
- Hours spent getting book to school per week
- Hours spent getting book home per week
- Hours spent with book in class per week



- Average size of class being taught from this book
- Average age of a maths teacher teaching from this book



- Number of pages in Grade 12 Maths textbook
- Number of pages in Grade 11 Maths textbook
- Number of pages in Grade 10 Maths textbook



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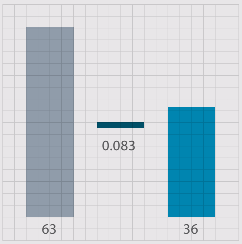
- Number of English volunteers who helped write this book



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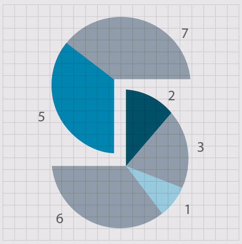

- Weekly UCT hackathons that contributed to this book
- Small office hackathons that contributed to this book
- Afrikaans hackathons that contributed to this book
- Virtual hackathons that contributed to this book

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EVERYTHING MATHS

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www.siyavula.com
info@siyavula.com
021 469 4771

Siyavula Authors

Alison Jenkin; Marina van Zyl; Luke Kannemeyer

Siyavula and DBE team

Dr Carl Scheffler; Bridget Nash; Ewald Zietsman; William Buthane Chauke; Leonard Gumani Mudau; Sthe Khanyile; Josephine Mamaroke Phatlane

Siyavula and Free High School Science Text contributors

Dr Mark Horner; Dr Samuel Halliday; Dr Sarah Blyth; Dr Rory Adams; Dr Spencer Wheaton

Iesrafeel Abbas; Sarah Abel; Taskeen Adam; Ross Adams; Tracey Adams; Dr Rory Adams; Andrea Africa; Wiehan Agenbag; Ismail Akhalwaya; Matthew Amundsen; Ben Anhalt; Prashant Arora; Bianca Bôhmer; Amos Baloyi; Bongani Baloyi; Raymond Barbour; Caro-Joy Barendse; Katie Barry; Dr Ilsa Basson; Richard Baxter; Tara Beckerling; Tim van Beek; Lisette de Beer; Jessie Bester; Mariaan Bester; Jennifer de Beyer; Dr Sarah Blyth; Sebastian Bodenstein; Martin Bongers; Dr Thinus Booysen; Ena Bosman; Janita Botha; Pieter Botha; Gareth Boxall; Stephan Brandt; Hannes Breytenbach; Alexander Briell; Wilbur Britz; Graeme Broster; Craig Brown; Michail Brynard; Richard Burge; Jan Buys; George Calder-Potts; Biddy Cameron; Eleanor Cameron; Mark Carolissen; Shane Carollisson; Richard Case; Sithembile Cele; Alice Chang; Faith Chaza; Richard Cheng; Fanny Cherblanc; Lizzy Chivaka; Dr Christine Chung; Dr Mareli Claasens; Brett Cocks; Zelmari Coetzee; Roché Compaan; Willem Conradie; Stefaan Conradie; Deanne Coppejans; Rocco Coppejans; Tim Craib; Dr Andrew Craig; Tim Crombie; Dan Crytser; Jock Currie; Dr Anne Dabrowski; Laura Daniels; Gareth Davies; Mia de; Tariq Desai; Sandra Dickson; Sean Dobbs; Buhle Donga; William Donkin; Esmi Dreyer; Matthew Duddy; Christel Durie; Fernando Durrell; Dr Dan Dwyer; Frans van Eeden; Kobus Ehlers; Alexander Ellis; Tom Ellis; Charl Esterhuysen; Andrew Fisher; Dr Philip Fourie; Giovanni Franzoni; Sanette Gildenhuys; Olivia Gillett; Ingrid von Glehn; Tamara von Glehn; Nicola Glenday; Lindsay Glesener; Kevin Godby; Dr Vanessa Godfrey; Terence Goldberg; Dr Johan Gonzalez; Saaligha Gool; Hemant Gopal; Dr Stephanie Gould; Umeshree Govender; Dr Ilse le Grange; Heather Gray; Lynn Greeff; Jaco Greyling; Martli Greyvenstein; Carine Grobbelaar; Suzanne Grové; Dr Tom Gutierrez; Brooke Haag; Kate Hadley; Alex Hall; Dr Sam Halliday; Asheena Hanuman; Dr Melanie Dymond Harper; Ebrahim Harris; Dr Nicholas Harrison; Neil Hart; Nicholas Hatcher; Jason Hayden; Laura Hayward; Dr William P. Heal; Pierre van Heerden; Dr Fritha Hennessy; Dr Colleen Henning; Anna Herrington; Shaun Hewitson; Millie Hilgart; Grant Hillebrand; Gregory Hingle; Nick Hobbs; Chris Holdsworth; Dr Benne Holwerda; Dr Mark Horner; Robert Hovden; Mfandaidza Hove; Jennifer Hsieh; George Hugo; Dr Belinda Huntley; Laura Huss; Prof Ed Jacobs; Hester Jacobs; Stefan Jacobs; Rowan Jelley;

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EVERYTHING MATHS

Mathematics is commonly thought of as being about numbers but mathematics is actually a language! Mathematics is the language that nature speaks to us in. As we learn to understand and speak this language, we can discover many of nature's secrets. Just as understanding someone's language is necessary to learn more about them, mathematics is required to learn about all aspects of the world – whether it is physical sciences, life sciences or even finance and economics.

The great writers and poets of the world have the ability to draw on words and put them together in ways that can tell beautiful or inspiring stories. In a similar way, one can draw on mathematics to explain and create new things. Many of the modern technologies that have enriched our lives are greatly dependent on mathematics. DVDs, Google searches, bank cards with PIN numbers are just some examples. And just as words were not created specifically to tell a story but their existence enabled stories to be told, so the mathematics used to create these technologies was not developed for its own sake, but was available to be drawn on when the time for its application was right.

There is in fact not an area of life that is not affected by mathematics. Many of the most sought after careers depend on the use of mathematics. Civil engineers use mathematics to determine how to best design new structures; economists use mathematics to describe and predict how the economy will react to certain changes; investors use mathematics to price certain types of shares or calculate how risky particular investments are; software developers use mathematics for many of the algorithms (such as Google searches and data security) that make programmes useful.

But, even in our daily lives mathematics is everywhere – in our use of distance, time and money. Mathematics is even present in art, design and music as it informs proportions and musical tones. The greater our ability to understand mathematics, the greater our ability to appreciate beauty and everything in nature. Far from being just a cold and abstract discipline, mathematics embodies logic, symmetry, harmony and technological progress. More than any other language, mathematics is everywhere and universal in its application.

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Well structured, impactful Corporate Social Investment (CSI) has the ability to contribute positively to nation building and drive positive change in the communities. MMI's commitment to social investment means that we are constantly looking for ways in which we can assist some of South Africa's most vulnerable citizens to expand their horizons and gain greater access to life's opportunities. This means that we do not view social investment as a nice to have or as an exercise in marketing or sponsorship but rather as a critical part of our contribution to society.

The merger between Metropolitan and Momentum was lauded for the complementary fit between two companies. This complementary fit is also evident in the focus areas of CSI programmes where Metropolitan and Momentum together cover and support the most important sectors and where the greatest need is in terms of social participation.

HIV/AIDS is becoming a manageable disease in many developed countries but in a country such as ours, it remains a disease where people are still dying of this scourge unnecessarily. Metropolitan continues to make a difference in making sure that HIV AIDS moves away from being a death sentence to a manageable disease. Metropolitan's other focus area is education which remains the key to economic prosperity for our country.

Momentum's focus on persons with disabilities ensures that this community is included and allowed to make their contribution to society. Orphaned and vulnerable children are another focus area for Momentum and projects supported ensure that children are allowed to grow up safely, to assume their role along with other children in inheriting a prosperous future.

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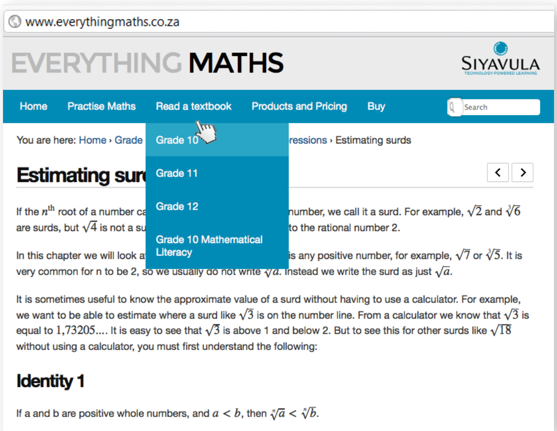

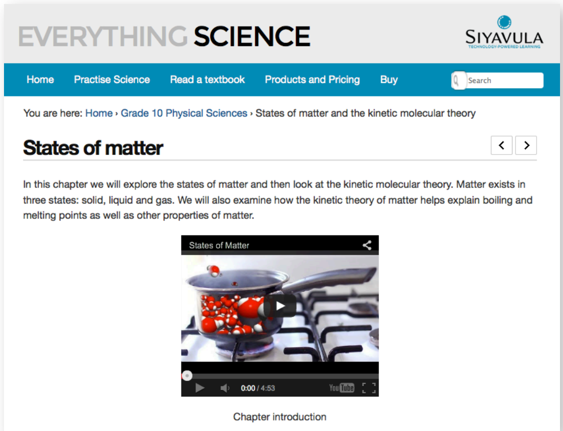

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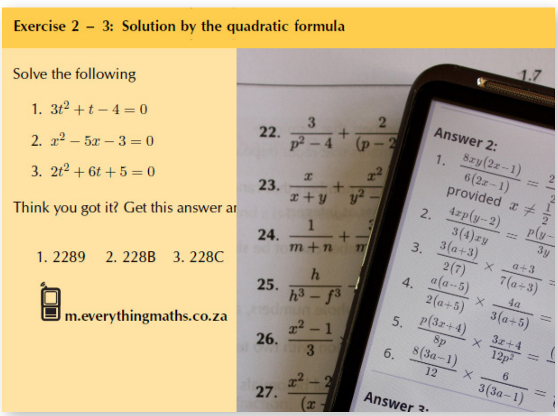

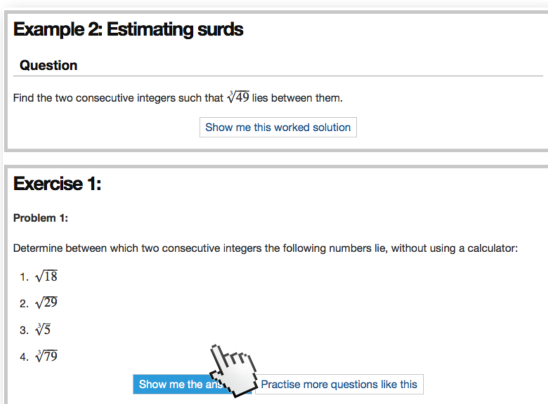

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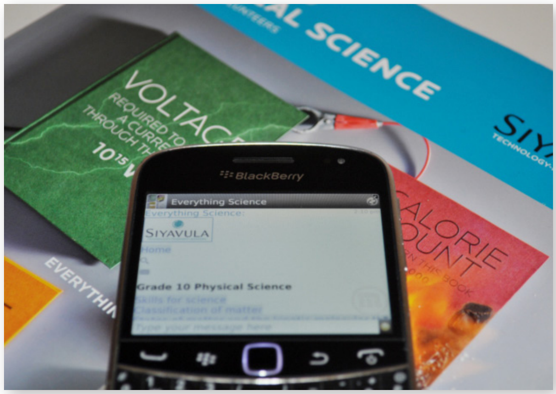

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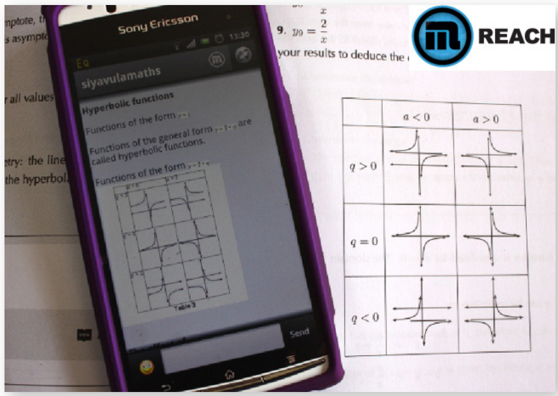

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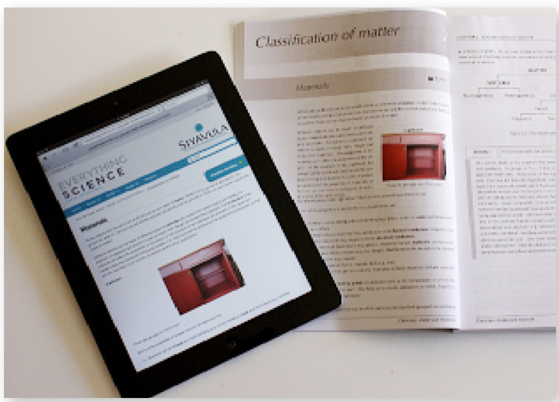

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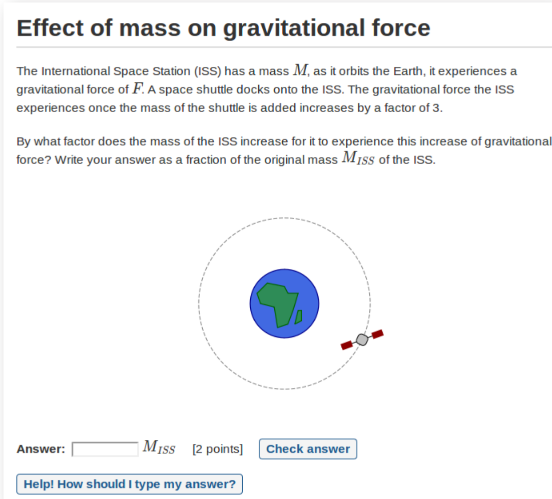

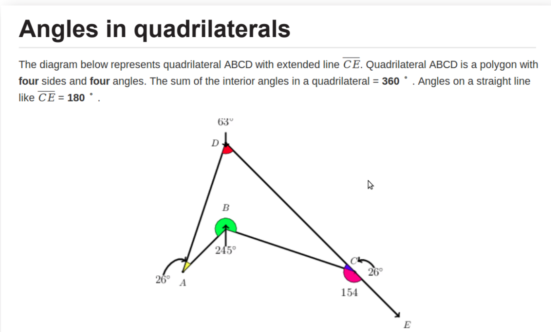

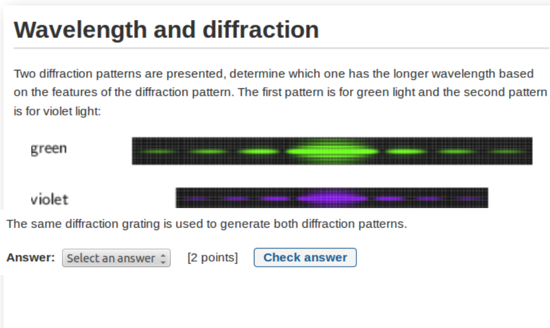

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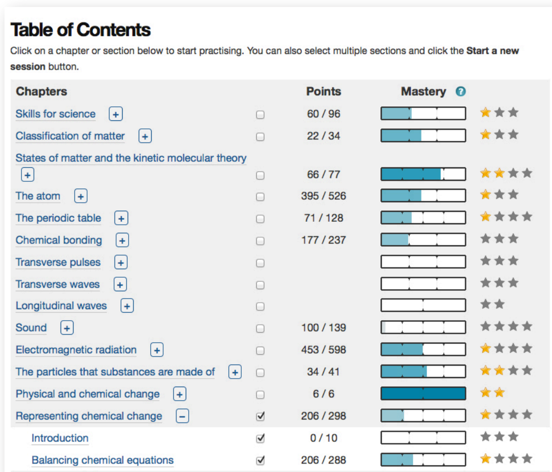

Contents

CHAPTER 0

Mathematics - Teachers guide

0.1 Blog posts

General blogs

- Educator's Monthly - Education News and Resources (http://www.teachersmonthly.com)
  - "We eat, breathe and live education! "
  - "Perhaps the most remarkable yet overlooked aspect of the South African teaching community is its enthusiastic, passionate spirit. Every day, thousands of talented, hard-working educators gain new insight from their work and come up with brilliant, inventive and exciting ideas. Educator's Monthly aims to bring educators closer and help them share knowledge and resources.
  - Our aim is twofold ...
    - To keep South African educators updated and informed.
    - To give educators the opportunity to express their views and cultivate their interests."
- Head Thoughts – Personal Reflections of a School Headmaster (http://headthoughts.co.za/)
  - blog by Arthur Preston
  - "Arthur is currently the headmaster of a growing independent school in Worcester, in the Western Cape province of South Africa. His approach to primary education is progressive and is leading the school through an era of new development and change."

Maths blog

- CEO: Circumspect Education Officer - Educating The Future
  - blog by Robyn Clark
  - "Mathematics teacher and inspirer."
  - http://clarkformaths.tumblr.com/
- dy/dan - Be less helpful
  - blog by Dan Meyer
  - "I'm Dan Meyer. I taught high school math between 2004 and 2010 and I am currently studying at Stanford University on a doctoral fellowship. My specific interests include curriculum design (answering the question, "how we design the ideal learning experience for students?") and teacher education (answering the questions, "how do teachers learn?" and "how do we retain more teachers?" and "how do we teach teachers to teach?")."
  - http://blog.mrmeyer.com
- Without Geometry, Life is Pointless - Musings on Math, Education, Teaching, and Research
  - blog by Avery
  - "I've been teaching some permutation (or is that combination?) of math and science to third through twelfth graders in private and public schools for 11 years. I'm also pursuing my EdD in education and will be both teaching and conducting research in my classroom this year."
  - http://mathteacherorstudent.blogspot.com/
- Overthinking my teaching - The Mathematics I Encounter in Classrooms
  - blog by Christopher Danielson

- "I think a lot about my math teaching. Perhaps too much. This is my outlet. I hope you find it interesting and that you'll let me know how it's going."
- http://christopherdanielson.wordpress.com

- A Recursive Process - Math Teacher Seeking Patterns
  - blog by Dan
  - "I am a High School math teacher in upstate NY. I currently teach Geometry, Computer Programming (Alice and Java), and two half year courses: Applied and Consumer Math. This year brings a new 21st century classroom (still not entirely sure what that entails) and a change over to standards based grades."
  - http://dandersod.wordpress.com
- Think Thank Thunk – Dealing with the Fear of Being a Boring Teacher
  - blog by Shawn Cornally
  - "I am Mr. Cornally. I desperately want to be a good teacher. I teach Physics, Calculus, Programming, Geology, and Bioethics. Warning: I have problem with using colons. I proof read, albeit poorly."
  - http://101studiostreet.com/wordpress/

0.2 Overview

Before 1994 there existed a number of education departments and subsequent curriculum according to the segregation that was so evident during the apartheid years. As a result, the curriculum itself became one of the political icons of freedom or suppression. Since then the government and political leaders have sought to try and develop one curriculum that is aligned with our national agenda of democratic freedom and equality for all, in fore-grounding the knowledge, skills and values our country believes our learners need to acquire and apply, in order to participate meaningfully in society as citizens of a free country. The National Curriculum Statement (NCS) of Grades R – 12 (DBE, 2012) therefore serves the purposes of:

- equipping learners, irrespective of their socio-economic background, race, gender, physical ability or intellectual ability, with the knowledge, skills and values necessary for self-fulfilment, and meaningful participation in society as citizens of a free country;
- providing access to higher education;
- facilitating the transition of learners from education institutions to the workplace; and
- providing employers with a sufficient profile of a learner's competencies.

Although elevated to the status of political icon, the curriculum remains a tool that requires the skill of an educator in interpreting and operationalising this tool within the classroom. The curriculum itself cannot accomplish the purposes outlined above without the community of curriculum specialists, material developers, educators and assessors contributing to and supporting the process, of the intended curriculum becoming the implemented curriculum. A curriculum can succeed or fail, depending on its implementation, despite its intended principles or potential on paper. It is therefore important that stakeholders of the curriculum are familiar with and aligned to the following principles that the NCS (CAPS) is based on:

Principle	Implementation
Social Transformation	Redressing imbalances of the past. Providing equal opportunities for all.
Active and Critical Learning	Encouraging an active and critical approach to learning. Avoiding excessive rote and uncritical learning of given truths.
High Knowledge and Skills	Learners achieve minimum standards of knowledge and skills specified for each grade in each subject.
Progression	Content and context shows progression from simple to complex.
Social and Environmental Justice and Human Rights	These practices as defined in the Constitution are infused into the teaching and learning of each of the subjects.
Valuing Indigenous Knowledge Systems	Acknowledging the rich history and heritage of this country.
Credibility, Quality and Efficiency	Providing an education that is globally comparable in quality.

This guide is intended to add value and insight to the existing National Curriculum for Grade 12 Mathematics, in line with its purposes and principles. It is hoped that this will assist you as the educator in optimising the implementation of the intended curriculum.

Curriculum requirements and objectives

The main objectives of the curriculum relate to the learners that emerge from our educational system. While educators are the most important stakeholders in the implementation of the intended curriculum, the quality of learner coming through this curriculum will be evidence of the actual attained curriculum from what was intended and then implemented.

These purposes and principles aim to produce learners that are able to:

- identify and solve problems and make decisions using critical and creative thinking;
- work effectively as individuals and with others as members of a team;
- organise and manage themselves and their activities responsibly and effectively;
- collect, analyse, organise and critically evaluate information;
- communicate effectively using visual, symbolic and/or language skills in various modes;
- use science and technology effectively and critically showing responsibility towards the environment and the health of others; and
- demonstrate an understanding of the world as a set of related systems by recognising that problem solving contexts do not exist in isolation.

The above points can be summarised as an independent learner who can think critically and analytically, while also being able to work effectively with members of a team and identify and solve problems through effective decision making. This is also the outcome of what educational research terms the "reformed" approach rather than the "traditional" approach many educators are more accustomed to. Traditional practices have their role and cannot be totally abandoned in favour of only reform practices. However, in order to produce more independent and mathematical thinkers, the reform ideology needs to be more embraced by educators within their instructional behaviour. Here is a table that can guide you to identify your dominant instructional practice and try to assist you in adjusting it (if necessary) to be more balanced and in line with the reform approach being suggested by the NCS (CAPS).

	Traditional Versus Reform Practices
Values	**Traditional** – values content, correctness of learners' responses and mathematical validity of methods. **Reform** – values finding patterns, making connections, communicating mathematically and problem-solving.
Teaching Methods	**Traditional** – expository, transmission, lots of drill and practice, step by step mastery of algorithms. **Reform** – hands-on guided discovery methods, exploration, modelling. High level reasoning processes are central.
Grouping Learners	**Traditional** – dominantly same grouping approaches. **Reform** – dominantly mixed grouping and abilities.

The subject of mathematics, by the nature of the discipline, provides ample opportunities to meet the reformed objectives. In doing so, the definition of mathematics needs to be understood and embraced by educators involved in the teaching and the learning of the subject. In research it has been well documented that, as educators, our conceptions of what mathematics is, has an influence on our approach to the teaching and learning of the subject.

Three possible views of mathematics can be presented. The instrumentalist view of mathematics assumes the stance that mathematics is an accumulation of facts, rules and skills that need to be used as a means to an end, without there necessarily being any relation between these components. The Platonist view of mathematics sees the subject as a static but unified body of certain knowledge, in which mathematics is discovered rather than created. The problem solving view of mathematics is a dynamic, continually expanding and evolving field of human creation and invention that is in itself a cultural product. Thus mathematics is viewed as a process of enquiry, not a finished product. The results remain constantly open to revision. It is suggested that a hierarchical order exists within these three views, placing the instrumentalist view at the lowest level and the problem solving view at the highest.

According to the NCS (CAPS):

Mathematics is the study of quantity, structure, space and change. Mathematicians seek out patterns, formulate new conjectures, and establish axiomatic systems by rigorous deduction from appropriately chosen axioms and definitions. Mathematics is a distinctly human activity practised by all cultures, for thousands of years. Mathematical problem solving enables us to understand the world (physical, social and economic) around us, and, most of all, to teach us to think creatively.

This corresponds well to the problem solving view of mathematics and may challenge some of our instrumentalist or Platonistic views of mathematics as a static body of knowledge of accumulated facts, rules and skills to be learnt and applied. The NCS (CAPS) is trying to discourage such an approach and encourage mathematics educators to dynamically and creatively involve their learners as mathematicians engaged in a process of study, understanding, reasoning, problem solving and communicating mathematically.

Below is a check list that can guide you in actively designing your lessons in an attempt to embrace the definition of mathematics from the NCS (CAPS) and move towards a problem solving conception of the subject. Adopting such an approach to the teaching and learning of mathematics will in turn contribute to the intended curriculum being properly implemented and attained through the quality of learners coming out of the education system.

Practice	Example
Learners engage in solving contextual problems related to their lives that require them to interpret a problem and then find a suitable mathematical solution.	Learners are asked to work out which bus service is the cheapest given the fares they charge and the distance they want to travel.
Learners engage in solving problems of a purely mathematical nature, which require higher order thinking and application of knowledge (non-routine problems).	Learners are required to draw a graph; they have not yet been given a specific technique on how to draw (for example a parabola), but have learnt to use the table method to draw straight-line graphs.
Learners are given opportunities to negotiate meaning.	Learners discuss their understanding of concepts and strategies for solving problems with each other and the educator.
Learners are shown and required to represent situations in various but equivalent ways (mathematical modelling).	Learners represent data using a graph, a table and a formula to represent the same data.
Learners individually do mathematical investigations in class, guided by the educator where necessary.	Each learner is given a paper containing the mathematical problem (for instance to find the number of prime numbers less than 50) that needs to be investigated and the solution needs to be written up. Learners work independently.
Learners work together as a group/team to investigate or solve a mathematical problem.	A group is given the task of working together to solve a problem that requires them investigating patterns and working through data to make conjectures and find a formula for the pattern.
Learners do drill and practice exercises to consolidate the learning of concepts and to master various skills.	Completing an exercise requiring routine procedures.
Learners are given opportunities to see the interrelatedness of the mathematics and to see how the different outcomes are related and connected.	While learners work through geometry problems, they are encouraged to make use of algebra.
Learners are required to pose problems for their educator and peer learners.	Learners are asked to make up an algebraic word problem (for which they also know the solution) for the person sitting next to them to solve.

Summary of topics and their relevance:

1. Functions – linear, quadratic, exponential, rational	Relevance
Introduce a more formal definition of a function and extend Grade 11 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Include linear, quadratic and some cubic polynomial functions, exponential and logarithmic functions, and some rational functions. The inverses of prescribed functions and be aware of the fact that, in the case of many-to-one functions, the domain has to be restricted if the inverse is to be a function. Problem solving and graph work involving the prescribed functions (including the logarithmic function).	Functions form a core part of learners' mathematical understanding and reasoning processes in algebra. This is also an excellent opportunity for contextual mathematical modelling questions.

2. Number Patterns, Sequences and Series	Relevance
Identify and solve problems involving number patterns that lead to arithmetic and geometric sequences and series, including infinite geometric series.	Much of mathematics revolves around the identification of patterns.

3. Finance, Growth and Decay	Relevance
Calculate the value of n in the formulae $A = P(1+i)^n$ and $A = P(1-i)^n$. Apply knowledge of geometric series to solve annuity and bond repayment problems. Critically analyse different loan options.	The mathematics of finance is very relevant to daily and long-term financial decisions learners will need to make in terms of investing, taking loans, saving and understanding exchange rates and their influence more globally.

4. Algebra	Relevance
Demonstrate an understanding of the definition of a logarithm and any laws needed to solve real life problems. Take note and understand, the Remainder and Factor Theorems for polynomials up to the third degree. Factorise third-degree polynomials (including examples which require the Factor Theorem).	Algebra provides the basis for mathematics learners to move from numerical calculations to generalising operations, simplifying expressions, solving equations and using graphs and inequalities in solving contextual problems.

5. Differential Calculus	Relevance
(a) An intuitive understanding of the concept of a limit. (b) Differentiation of specified functions from first principles. (c) Use of the specified rules of differentiation. (d) The equations of tangents to graphs. (e) The ability to sketch graphs of cubic functions. (f) Practical problems involving optimization and rates of change (including the calculus of motion).	The central aspect of rate of change to differential calculus is a basis to further understanding of limits, gradients and calculations and formulae necessary for work in engineering fields, e.g. designing roads, bridges etc.

6. Probability	Relevance
(a) Generalisation of the fundamental counting principle. (b) Probability problems using the fundamental counting principle.	This topic is helpful in developing good logical reasoning in learners and for educating them in terms of real-life issues such as gambling and the possible pitfalls thereof.

7. Euclidean Geometry and Measurement	Relevance
(a) Revise earlier (Grade 9) work on the necessary and sufficient conditions for polygons to be similar. (b) Prove (accepting results established in earlier grades): • that a line drawn parallel to one side of a triangle divides the other two sides proportionally (and the Midpoint Theorem as a special case of this theorem); • that equiangular triangles are similar; • that triangles with sides in proportion are similar; • the Pythagorean Theorem by similar triangles; and • riders.	The thinking processes and mathematical skills of proving conjectures and identifying false conjectures is more the relevance here than the actual content studied. The surface area and volume content studied in real-life contexts of designing kitchens, tiling and painting rooms, designing packages, etc. is relevant to the current and future lives of learners.

8. Trigonometry	Relevance
Proof and use of the compound angle and double angle identities Solve problems in two and three dimensions.	Trigonometry has several uses within society, including within navigation, music, geographical locations and building design and construction.

9. Analytical Geometry	Relevance
Use a two-dimensional Cartesian coordinate system to derive and apply: • the equation of a circle (any centre); and • the equation of a tangent to a circle at a given point on the circle.	This section provides a further application point for learners' algebraic and trigonometric interaction with the Cartesian plane. Artists and design and layout industries often draw on the content and thought processes of this mathematical topic.

10. Statistics	Relevance
(a) Represent bivariate numerical data as a scatter plot and suggest intuitively and by simple investigation whether a linear, quadratic or exponential function would best fit the data. (b) Use a calculator to calculate the linear regression line which best fits a given set of bivariate numerical data. (c) Use a calculator to calculate the correlation co-efficient of a set of bivariate numerical data and make relevant deductions.	Citizens are daily confronted with interpreting data presented from the media. Often this data may be biased or misrepresented within a certain context. In any type of research, data collection and handling is a core feature. This topic also educates learners to become more socially and politically educated with regards to the media.

Mathematics educators also need to ensure that the following important specific aims and general principles are applied in mathematics activities across all grades:

- Calculators should only be used to perform standard numerical computations and verify calculations done by hand.
- Real-life problems should be incorporated into all sections to keep mathematical modelling as an important focal point of the curriculum.
- Investigations give learners the opportunity to develop their ability to be more methodical, to generalise and to make and justify and/or prove conjectures.
- Appropriate approximation and rounding skills should be taught and continuously included and encouraged in activities.
- The history of mathematics should be incorporated into projects and tasks where possible, to illustrate the human aspect and developing nature of mathematics.
- Contextual problems should include issues relating to health, social, economic, cultural, scientific, political and environmental issues where possible.
- Conceptual understanding of when and why should also feature in problem types.
- Mixed ability teaching requires educators to challenge able learners and provide remedial support where necessary.
- Misconceptions exposed by assessment need to be dealt with and rectified by questions designed by educators.
- Problem solving and cognitive development should be central to all mathematics teaching and learning so that learners can apply the knowledge effectively.

Allocation of teaching time:

Time allocation for Mathematics per week: 4 hours and 30 minutes e.g. six forty-five minute periods per week.

Term	Topic	No. of weeks
Term 1	Patterns, sequences and series	3
	Functions and inverses	3
	Exponential and logarithmic functions	1
	Finance, growth and decay	2
	Trigonometry - compound angles	2
Term 2	Trigonometry 2D and 3D applications	2
	Polynomial functions	1
	Differential calculus	3
	Analytical geometry	2
	Mid-year exams	3
Term 3	Euclidean geometry	2
	Statistics	2
	Probability	2
	Revision	1
	Trial exams	3
Term 4	Revision	3
	Final exams	6

Please see page 20 of the Curriculum and Assessment Policy Statement for the sequencing and pacing of topics.

0.3 Assessment

"Educator assessment is part of everyday teaching and learning in the classroom. Educators discuss with learners, guide their work, ask and answer questions, observe, help, encourage and challenge. In addition, they mark and review written and other kinds of work. Through these activities they are continually finding out about their learners' capabilities and achievements. This knowledge then informs plans for future work. It is this continuous process that makes up educator assessment. It should not be seen as a separate activity necessarily requiring the use of extra tasks or tests."

As the quote above suggests, assessment should be incorporated as part of the classroom practice, rather than as a separate activity. Research during the past ten years indicates that learners get a sense of what they do and do not know, what they might do about this and how they feel about it, from frequent and regular classroom assessment and educator feedback. The educator's perceptions of and approach to assessment (both formal and informal assessment) can have an influence on the classroom culture that is created with regard to the learners' expectations of and performance in assessment tasks. Literature on classroom assessment distinguishes between two different purposes of assessment; assessment of learning and assessment for learning.

Assessment of learning tends to be a more formal assessment and assesses how much learners have learnt or understood at a particular point in the annual teaching plan. The NCS (CAPS) provides comprehensive guidelines on the types of and amount of formal assessment that needs to take place within the teaching year to make up the school-based assessment mark. The school-based assessment mark contributes 25% of the final percentage of a learner's promotion mark, while the end-of-year examination constitutes the other 75% of the annual promotion mark. Learners are expected to have 7 formal assessment tasks for their school-based assessment mark. The number of tasks and their weighting in the Grade 12 Mathematics curriculum is summarised below:

		Tasks	**Weight (percent)**
School-Based Assessment	Term 1	Test Project/Investigation Assignment	10 20 10
	Term 2	Test Mid-Year Examination	10 15
	Term 3	Test Trial Examination	10 25
	Term 4		
School-Based Assessment Mark			100
School-Based Assessment Mark (as a percent of Promotion Mark)			25%
End-of-Year Examination			75%
Promotion Mark			100%

The following provides a brief explanation of each of the assessment tasks included in the assessment programme above.

Tests

All mathematics educators are familiar with this form of formal assessment. Tests include a variety of items/questions covering the topics that have been taught prior to the test. The new NCS (CAPS) also stipulates that mathematics tests should include questions that cover the following four types of cognitive levels in the stipulated weightings:

Cognitive levels	Description	Weighting (percent)
Knowledge	Estimation and appropriate rounding of numbers. Proofs of prescribed theorems. Derivation of formulae. Straight recall. Identification and direct use of formula on information sheet (no changing of the subject). Use of mathematical facts. Appropriate use of mathematical vocabulary.	20
Routine procedures	Perform well known procedures. Simple applications and calculations. Derivation from given information. Identification and use (including changing the subject) of correct formula. Questions generally similar to those done in class.	35
Complex procedures	Problems involve complex calculations and/or higher reasoning. There is often not an obvious route to the solution. Problems need not be based on real world context. Could involve making significant connections between different representations. Require conceptual understanding.	30
Problem solving	Unseen, non-routine problems (which are not necessarily difficult). Higher order understanding and processes are often involved. Might require the ability to break the problem down into its constituent parts.	15

The breakdown of the tests over the four terms is summarised from the NCS (CAPS) assessment programme as follows:

Term 1: One test/assignment (of at least 50 marks and one hour).

Term 2: One test (of at least 50 marks and one hour).

Term 3: One test (of at least 50 marks and one hour).

Term 4: None.

Projects/Investigations

Investigations and projects consist of open-ended questions that initiate and expand thought processes. Acquiring and developing problem-solving skills are an essential part of doing investigations and projects. These tasks provide learners with the opportunity to investigate, gather information, tabulate results, make conjectures and justify or prove these conjectures. Examples of investigations and projects and possible marking rubrics are provided in the next section on assessment support. The NCS (CAPS) assessment programme indicates that only one project or investigation (of at least 50 marks) should be included per year. Although the project/investigation is scheduled in the assessment programme for the first term, it could also be done in the second term.

Assignments

The NCS (CAPS) includes the following tasks as good examples of assignments:

- Open book test
- Translation task
- Error spotting and correction
- Shorter investigation
- Journal entry
- Mind-map (also known as a metacog)
- Olympiad (first round)
- Mathematics tutorial on an entire topic
- Mathematics tutorial on more complex/problem solving questions

The NCS (CAPS) assessment programme requires one assignment in term 1 (of at least 50 marks) which could also be a combination of some of the suggested examples above. More information on these suggested examples of assignments and possible rubrics are provided in the following section on assessment support.

Examinations

Educators are also all familiar with this summative form of assessment that is usually completed twice a year: mid-year examinations and end-of-year examinations. These are similar to the tests but cover a wider range of topics completed prior to each examination. The NCS (CAPS) stipulates that each examination should also cover the four cognitive levels according to their recommended weightings as summarised in the section above on tests. The following table summarises the requirements and information from the NCS (CAPS) for the two examinations.

Examination	Marks	Breakdown	Content and Mark distribution
Mid-Year Exams	300 150 + 150	Either mid-year or trial exams must consist of two 3 hour papers.	Topics completed
Trial Exams	300	Either mid-year or trial exams must consist of two 3 hour papers.	All topics
End-of-Year Exams	150 +	Paper 1: 3 hours	Patterns and sequences (± 25) Finance, growth and decay (± 15) Functions and graphs (± 35) Algebra and equations (± 25) Calculus (± 35) Probability (± 15)
End-of-Year Exams	150	Paper 2: 3 hours	Euclidean geometry and measurement (± 50) Analytical geometry (± 40) Statistics (± 20) Trigonometry (± 40)

In the annual teaching plan summary of the NCS (CAPS) in Mathematics for Grade 12, the pace setter section provides a detailed model of the suggested topics to be covered each week of each term and the accompanying formal assessment.

Assessment **for** learning tends to be more informal and focuses on using assessment in and of daily classroom activities that can include:

1. Marking homework
2. Baseline assessments
3. Diagnostic assessments
4. Group work
5. Class discussions
6. Oral presentations
7. Self-assessment
8. Peer-assessment

These activities are expanded on in the next section on assessment support and suggested marking rubrics are provided. Where formal assessment tends to restrict the learner to written assessment tasks, the informal assessment is necessary to evaluate and encourage the progress of the learners in their verbal mathematical reasoning and communication skills. It also provides a less formal assessment environment that allows learners to openly and honestly assess themselves and each other, taking responsibility for their own learning, without the heavy weighting of the performance (or mark) component. The assessment for learning tasks should be included in the classroom activities at least once a week (as part of a lesson) to ensure that the educator is able to continuously evaluate the learners' understanding of the topics covered as well as the effectiveness, and identify any possible deficiencies in his or her own teaching of the topics.

Assessment support

A selection of explanations, examples and suggested marking rubrics for the assessment of learning (formal) and the assessment for learning (informal) forms of assessment discussed in the preceding section are provided in this section.

Baseline assessment

Baseline assessment is a means of establishing:

- What prior knowledge a learner possesses
- What the extent of knowledge is that they have regarding a specific learning area?
- The level they demonstrate regarding various skills and applications
- The learner's level of understanding of various learning areas

It is helpful to educators in order to assist them in taking learners from their individual point of departure to a more advanced level and to thus make progress. This also helps avoid large ″gaps″ developing in the learners' knowledge as the learner moves through the education system. Outcomes-based education is a more learner-centered approach than we are used to in South Africa, and therefore the emphasis should now be on the level of each individual learner rather than that of the whole class.

The baseline assessments also act as a gauge to enable learners to take more responsibility for their own learning and to view their own progress. In the traditional assessment system, the weaker learners often drop from a 40% average in the first term to a 30% average in the fourth term due to an increase in workload, thus demonstrating no obvious progress. Baseline assessment, however, allows for an initial assigning of levels which can be improved upon as the learner progresses through a section of work and shows greater knowledge, understanding and skill in that area.

Diagnostic assessments

These are used to specifically find out if any learning difficulties or problems exist within a section of work in order to provide the learner with appropriate additional help and guidance.

The assessment helps the educator and the learner identify problem areas, misunderstandings, misconceptions and incorrect use and interpretation of notation.

Some points to keep in mind:

- Try not to test too many concepts within one diagnostic assessment.
- Be selective in the type of questions you choose.
- Diagnostic assessments need to be designed with a certain structure in mind. As an educator, you should decide exactly what outcomes you will be assessing and structure the content of the assessment accordingly.
- The assessment is marked differently to other tests in that the mark is not the focus but rather the type of mistakes the learner has made.

An example of an understanding rubric for educators to record results is provided below:

0: indicates that the learner has not grasped the concept at all and that there appears to be a fundamental mathematical problem.

1: indicates that the learner has gained some idea of the content, but is not demonstrating an understanding of the notation and concept.

2: indicates evidence of some understanding by the learner but further consolidation is still required.

3: indicates clear evidence that the learner has understood the concept and is using the notation correctly.

Calculator worksheet - diagnostic skills assessment

1. Calculate:
   a) $242 + 63 =$
   b) $2 - 36 \times (114 + 25) =$
   c) $\sqrt{144 + 25} =$
   d) $\sqrt[4]{729} =$
   e) $-312 + 6 + 879 - 321 + 18\ 901 =$
2. Calculate:
   a) $\frac{2}{7} + \frac{1}{3} =$
   b) $2\frac{1}{5} - \frac{2}{9} =$
   c) $-2\frac{5}{6} + \frac{3}{8} =$
   d) $4 - \frac{3}{4} \times \frac{5}{7} =$
   e) $\left(\frac{9}{10} - \frac{8}{9}\right) \div \frac{3}{5} =$
   f) $2 \times \left(\frac{4}{5}\right)^2 - \left(\frac{19}{25}\right) =$
   g) $\sqrt{\frac{9}{4} - \frac{4}{16}} =$

Self-Assesment Rubric:

Name:

Question	Answer	$\surd$	X	If X, write down sequence of keys pressed
1a				
1b				
1c				
1d				
1e				
Subtotal				
2a				
2b				
2c				
2d				
2e				
Subtotal				
Total				

Educator Assessment Rubric:

Type of skill	Competent	Needs practice	Problem
Raising to a power			
Finding a root			
Calculations with Fractions			
Brackets and order of operations			
Estimation and mental control			

Guidelines for Calculator Skills Assessment:

Type of skill	Sub-Division	Questions
Raising to a Power	Squaring and cubing Higher order powers	1a, 2f 1b
Finding a Root	Square and cube roots Higher order roots	1c, 2g 1d
Calculations with Fractions	Basic operations Mixed numbers Negative numbers Squaring fractions Square rooting fractions	2a, 2d 2b, 2c 1e, 2c 2f 2g
Brackets and Order of Operations	Correct use of brackets or order of operations	1b, 1c, 2e, 2f, 2g
Brackets and Order of Operations	Estimation and Mental Control	All

Suggested guideline to allocation of overall levels

Level 1

- Learner is able to do basic operations on calculator.
- Learner is able to do simple calculations involving fractions.
- Learner does not display sufficient mental estimation and control techniques.

Level 2

- Learner is able to do basic operations on calculator.
- Learner is able to square and cube whole numbers as well as find square and cube roots of numbers.
- Learner is able to do simple calculations involving fractions as well as correctly execute calculations involving mixed numbers.
- Learner displays some degree of mental estimation awareness.

Level 3

- Learner is able to do basic operations on calculator.
- Learner is able to square and cube rational numbers as well as find square and cube roots of numbers.
- Learner is also able to calculate higher order powers and roots.
- Learner is able to do simple calculations involving fractions as well as correctly execute calculations involving mixed numbers.
- Learner works correctly with negative numbers.
- Learner is able to use brackets in certain calculations but has still not fully understood the order of operations that the calculator has been programmed to execute, hence the need for brackets.
- Learner is able to identify possible errors and problems in their calculations but needs assistance solving the problem.

Level 4

- Learner is able to do basic operations on calculator.
- Learner is able to square and cube rational numbers as well as find square and cube roots.
- Learner is also able to calculate higher order powers and roots.
- Learner is able to do simple calculations involving fractions as well as correctly execute calculations involving mixed numbers.
- Learner works correctly with negative numbers.
- Learner is able to work with brackets correctly and understands the need and use of brackets and the "= key" in certain calculations due to the nature of a scientific calculator.
- Learner is able to identify possible errors and problems in their calculations and to find solutions to these in order to arrive at a "more viable" answer.

Other short diagnostic tests

These are short tests that assess small quantities of recall knowledge and application ability on a day-to-day basis. Such tests could include questions on one or a combination of the following:

- Definitions
- Theorems
- Riders (geometry)
- Formulae
- Applications
- Combination questions

Exercises

This entails any work from the textbook or other source that is given to the learner, by the educator, to complete either in class or at home. Educators should encourage learners not to copy each other's work and be vigilant when controlling this work. It is suggested that such work be marked/controlled by a check list (below) to speed up the process for the educator.

The marks obtained by the learner for a specific piece of work need not be based on correct and/or incorrect answers but preferably on the following:

1. the effort of the learner to produce answers.
2. the quality of the corrections of work that was previously incorrect.
3. the ability of the learner to explain the content of some selected examples (whether in writing or orally).

The following rubric can be used to assess exercises done in class or as homework:

Criteria	Performance indicators		
Work Done	2 All the work	1 Partially completed	0 No work done
Work Neatly Done	2 Work neatly done	1 Some work not neatly done	0 Messy and muddled
Corrections Done	2 All corrections done consistently	1 At least half of the corrections done	0 No corrections done
Correct Mathematical Method	2 Consistently	1 Sometimes	0 Never
Understanding of Mathematical Techniques and Processes	2 Can explain concepts and processes precisely	1 Explanations are ambiguous or not focused	0 Explanations are confusing or irrelevant

Journal entries

A journal entry is an attempt by a learner to express in the written word what is happening in Mathematics. It is important to be able to articulate a mathematical problem, and its solution in the written word.

This can be done in a number of different ways:

- Today in Maths we learnt...
- Write a letter to a friend, who has been sick, explaining what was done in class today.
- Explain the thought process behind trying to solve a particular maths problem, e.g. sketch the graph of $y = x^2 - 2x^2 + 1$ and explain how to sketch such a graph.
- Give a solution to a problem, decide whether it is correct and if not, explain the possible difficulties experienced by the person who wrote the incorrect solution.

A journal is an invaluable tool that enables the educator to identify any mathematical misconceptions of the learners. The marking of this kind of exercise can be seen as subjective but a marking rubric can simplify the task.

The following rubric can be used to mark journal entries. The learners must be given the marking rubric before the task is done.

Task	Competent (2 marks)	Still developing (1 mark)	Not yet developed (0 marks)
Completion in time limit?			
Correctness of the explanation?			
Correct and relevant use of mathematical language?			
Has the concept been interpret correctly?			

Translations

Translations assess the learner's ability to translate from words into mathematical notation or to give an explanation of mathematical concepts in words. Often when learners can use mathematical language and notation correctly, they demonstrate a greater understanding of the concepts.

For example:

Write the letter of the correct expression next to the matching number:

x increased by 10	a)	xy
The product of x and y	b)	x^2
The sum of a certain number and	c)	x^2
double that number	d)	$29x$
Half of a certain number multiplied by itself	e)	$\frac{1}{2} \times 2$
Two less than x	f)	$x + x + 2$
A certain number multiplied by itself	g)	x^2

Group work

One of the principles in the NCS (CAPS) is to produce learners who are able to work effectively within a group. Learners generally find this difficult to do. Learners need to be encouraged to work within small groups. Very often it is while learning under peer assistance that a better understanding of concepts and processes is reached. Clever learners usually battle with this sort of task, and yet it is important that they learn how to assist and communicate effectively with other learners.

Mind maps or metacogs

A metacog or "mind map" is a useful tool. It helps to associate ideas and make connections that would otherwise be too unrelated to be linked. A metacog can be used at the beginning or end of a section of work in order to give learners an overall perspective of the work covered, or as a way of recalling a section already completed. It must be emphasised that it is not a summary. Whichever way you use it, it is a way in which a learner is given the opportunity of doing research in a particular field and can show that he/she has an understanding of the required section.

This is an open book form of assessment and learners may use any material they feel will assist them. It is suggested that this activity be practised, using other topics, before a test metacog is submitted for portfolio assessment purposes.

On completion of the metacog, learners must be able to answer insightful questions on the metacog. This is what sets it apart from being just a summary of a section of work. Learners must refer to their metacog when answering the questions, but may not refer to any reference material. Below are some guidelines to give to learners to adhere to when constructing a metacog as well as two examples to help you get learners started. A marking rubric is also provided. This should be made available to learners before they start constructing their metacogs. On the next page is a model question for a metacog, accompanied by some sample questions that can be asked within the context of doing a metacog about analytical geometry.

A basic metacog is drawn in the following way:

- Write the title/topic of the subject in the centre of the page and draw a circle around it.
- For the first main heading of the subject, draw a line out from the circle in any direction, and write the heading above or below the line.
- For sub-headings of the main heading, draw lines out from the first line for each subheading and label each one.
- For individual facts, draw lines out from the appropriate heading line.

Metacogs are one's own property. Once a person understands how to assemble the basic structure they can develop their own coding and conventions to take things further, for example to show linkages between facts. The following suggestions may assist educators and learners to enhance the effectiveness of their metacogs:

- Use single words or simple phrases for information. Excess words just clutter the metacog and take extra time to write down.
- Print words – joined up or indistinct writing can be more difficult to read and less attractive to look at.
- Use colour to separate different ideas – this will help your mind separate ideas where it is necessary, and helps visualisation of the metacog for easy recall. Colour also helps to show organisation.
- Use symbols and images where applicable. If a symbol means something to you, and conveys more information than words, use it. Pictures also help you to remember information.
- Use shapes, circles and boundaries to connect information – these are additional tools to help show the grouping of information.

Use the concept of analytical geometry as your topic and construct a mind map (or metacog) containing all the information (including terminology, definitions, formulae and examples) that you know about the topic of analytical geometry.

Possible questions to ask the learner on completion of their metacog:

- Briefly explain to me what the mathematics topic of analytical geometry entails.
- Identify and explain the distance formula, the derivation and use thereof for me on your metacog.
- How does the calculation of gradient in analytical geometry differ (or not) from the approach used to calculate gradient in working with functions?

Here is a suggested simple rubric for marking a metacog:

Task	**Competent (2 Marks)**	**Still Developing (1 Mark)**	**Not Yet Developed (1 Mark)**
Completion in Time Limit			
Main Headings			
Correct Theory (Formulae, Definitions, Terminology etc.)			
Explanation			
Readability			

10 marks for the questions, which are marked using the following scale:

0 - no attempt or a totally incorrect attempt has been made

1 - a correct attempt was made, but the learner did not get the correct answer

2 - a correct attempt was made and the answer is correct

Investigations

Investigations consist of open-ended questions that initiate and expand thought processes. Acquiring and developing problem-solving skills are an essential part of doing investigations.

It is suggested that 2 – 3 hours be allowed for this task. During the first 30 – 45 minutes learners could be encouraged to talk about the problem, clarify points of confusion, and discuss initial conjectures with others. The final written-up version should be done individually though and should be approximately four pages.

Assessing investigations may include feedback/ presentations from groups or individuals on the results keeping the following in mind:

- following of a logical sequence in solving the problems
- pre-knowledge required to solve the problem
- correct usage of mathematical language and notation
- purposefulness of solution
- quality of the written and oral presentation

Some examples of suggested marking rubrics are included on the next few pages, followed by a selection of topics for possible investigations.

The following guidelines should be provided to learners before they begin an investigation:

General Instructions Provided to Learners

- You may choose any one of the projects/investigations given (see model question on investigations)
- You should follow the instructions that accompany each task as these describe the way in which the final product must be presented.
- You may discuss the problem in groups to clarify issues, but each individual must write-up their own version.
- Copying from fellow learners will cause the task to be disqualified.
- Your educator is a resource to you, and though they will not provide you with answers / solutions, they may be approached for hints.

The investigation is to be handed in on the due date, indicated to you by your educator. It should have as a minimum:

- A description of the problem.
- A discussion of the way you set about dealing with the problem.
- A description of the final result with an appropriate justification of its validity.
- Some personal reflections that include mathematical or other lessons learnt, as well as the feelings experienced whilst engaging in the problem.
- The written-up version should be attractively and neatly presented on about four A4 pages.
- Whilst the use of technology is encouraged in the presentation, the mathematical content and processes must remain the major focus.

Below is an example of a possible rubric to use when marking investigations:

Level of Performance	Criteria
4	• Contains a complete response. • Clear, coherent, unambiguous and elegant explanation. • Includes clear and simple diagrams where appropriate. • Shows understanding of the question's mathematical ideas and processes. • Identifies all the important elements of the question. • Includes examples and counter examples. • Gives strong supporting arguments. • Goes beyond the requirements of the problem.
3	• Contains a complete response. • Explanation less elegant, less complete. • Shows understanding of the question's mathematical ideas and processes. • Identifies all the important elements of the question. • Does not go beyond the requirements of the problem.
2	• Contains an incomplete response. • Explanation is not logical and clear. • Shows some understanding of the question's mathematical ideas and processes. • Identifies some of the important elements of the question. • Presents arguments, but incomplete. • Includes diagrams, but inappropriate or unclear.
1	• Contains an incomplete response. • Omits significant parts or all of the question and response. • Contains major errors. • Uses inappropriate strategies.
0	• No visible response or attempt

Orals

An oral assessment involves the learner explaining to the class as a whole, a group or the educator his or her understanding of a concept, a problem or answering specific questions. The focus here is on the correct use of mathematical language by the learner and the conciseness and logical progression of their explanation as well as their communication skills.

Orals can be done in a number of ways:

- A learner explains the solution of a homework problem chosen by the educator.
- The educator asks the learner a specific question or set of questions to ascertain that the learner understands, and assesses the learner on their explanation.
- The educator observes a group of learners interacting and assesses the learners on their contributions and explanations within the group.
- A group is given a mark as a whole, according to the answer given to a question by any member of a group.

An example of a marking rubric for an oral:

1 - the learner has understood the question and attempts to answer it

2 - the learner uses correct mathematical language

2 - the explanation of the learner follows a logical progression

2 - the learner's explanation is concise and accurate

2 - the learner shows an understanding of the concept being explained

1 - the learner demonstrates good communication skills

Maximum mark $= 10$

An example of a peer-assessment rubric for an oral:

My name:

Name of person I am assessing:

Criteria	**Mark Awarded**	**Maximum Mark**
Correct Answer		2
Clarity of Explanation		3
Correctness of Explanation		3
Evidence of Understanding		2
Total		10

CHAPTER 1

Sequences and series

- Discuss and explain important terminology.
- Be consistent with the use of "common" difference and "constant" ratio to avoid confusing learners.
- Learners must understand the difference between arithmetic and geometric sequences.
- Explain sigma notation carefully as many learners have difficulty with this concept.
- Encourage learners to use the correct notation (for example T_n, S_n etc.) when solving problems.
- Use the investigation for the sum of an infinite series to introduce the concept of convergence and divergence.

1.1 Arithmetic sequences

Exercise 1 – 1: Arithmetic sequences

Find the common difference and write down the next 3 terms of the sequence.

1. $2; 6; 10; 14; 18; 22; \ldots$

   Solution:

$$\begin{aligned} d &= 6 - 2 \\ &= 4 \\ \text{or} & \\ d &= 10 - 6 \\ &= 4 \\ \therefore T_7 &= 22 + 4 \\ &= 26 \\ T_8 &= 26 + 4 \\ &= 30 \\ T_9 &= 30 + 4 \\ &= 34 \end{aligned}$$

2. $-1; -4; -7; -10; -13; -16; \ldots$

   Solution:

$$\begin{aligned} d &= -4 - (-1) \\ &= -3 \\ \text{or} & \\ d &= -7 - (-4) \\ &= -3 \\ \therefore T_7 &= -16 - 3 \\ &= -19 \\ T_8 &= -19 - 3 \\ &= -22 \\ T_9 &= -22 - 3 \\ &= -25 \end{aligned}$$

3. $-5; -3; -1; 1; 3; \ldots$

Solution:

$$\begin{aligned} d &= -3-(-5)\\ &= 2\\ \text{or}&\\ d &= -1-(-3)\\ &= 2\\ \therefore T_7 &= 3+2\\ &= 5\\ T_8 &= 5+2\\ &= 7\\ T_9 &= 7+2\\ &= 9 \end{aligned}$$

4. $-1; 10; 21; 32; 43; 54; \ldots$

Solution:

$$\begin{aligned} d &= 10-(-1)\\ &= 11\\ \text{or}&\\ d &= 21-10\\ &= 11\\ \therefore T_7 &= 54+11\\ &= 65\\ T_8 &= 65+11\\ &= 76\\ T_9 &= 76+11\\ &= 87 \end{aligned}$$

5. $a-3b; a-b; a+b; a+3b; \ldots$

Solution:

This is an example of a non-numeric arithmetic sequence.

$$\begin{aligned} d &= (a-b)-(a-3b)\\ &= a-b-a+3b\\ &= 2b\\ \text{or}&\\ d &= (a+b)-(a-b)\\ &= a+b-a+b\\ &= 2b\\ \therefore T_7 &= a+3b+2b\\ &= a+5b\\ T_8 &= a+5b+2b\\ &= a+7b\\ T_9 &= a+7b+2b\\ &= a+9b \end{aligned}$$

6. $-2; -\frac{3}{2}; -1; -\frac{1}{2}; 0; \frac{1}{2}; 1; \ldots$

Solution:

$$\begin{aligned} d &= -\frac{3}{2} - (-2) \\ &= \frac{1}{2} \end{aligned}$$

or

$$\begin{aligned} d &= -1 - \left(-\frac{3}{2}\right) \\ &= \frac{1}{2} \\ \therefore T_8 &= 1 + \frac{1}{2} \\ &= \frac{3}{2} \\ T_9 &= \frac{3}{2} + \frac{1}{2} \\ &= 2 \\ T_{10} &= 2 + \frac{1}{2} \\ &= \frac{5}{2} \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.
1. 284H 2. 284J 3. 284K 4. 284M 5. 284N 6. 284P

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The general term for an arithmetic sequence

Exercise 1 – 2: Arithmetic Sequences

1. Given the sequence $7; 5{,}5; 4; 2{,}5; \ldots$

   a) Find the next term in the sequence.
   Solution:

$$\begin{aligned} d &= 5{,}5 - 7 \\ &= -1{,}5 \\ \therefore T_5 &= 2{,}5 + (-1{,}5) \\ &= 1 \end{aligned}$$

   b) Determine the general term of the sequence.
   Solution:

$$\begin{aligned} T_n &= a + (n-1)d \\ &= 7 + (n-1)(-1{,}5) \\ &= 8{,}5 - 1{,}5n \end{aligned}$$

   c) Which term has a value of -23?
   Solution:

$$\begin{aligned} T_n &= 8{,}5 - 1{,}5n \\ \therefore -23 &= 8{,}5 - 1{,}5n \\ 31{,}5 &= 1{,}5n \\ \therefore n &= 21 \\ \text{Therefore } T_{21} &= -23 \end{aligned}$$

2. Given the sequence $2; 6; 10; 14; \ldots$

   a) Is this an arithmetic sequence? Justify your answer by calculation.
   Solution:

$$\begin{aligned} T_2 - T_1 &= 6 - 2 = 4 \\ T_3 - T_2 &= 10 - 6 = 4 \\ T_4 - T_3 &= 14 - 10 = 4 \end{aligned}$$

   Yes, this is an arithmetic sequence since there is a common difference of 4 between consecutive terms.

   b) Calculate T_{55}.
   Solution:

$$\begin{aligned} T_n &= a + (n-1)d \\ &= 2 + (n-1)4 \\ &= 4n - 2 \\ \therefore T_{55} &= 4(55) - 2 \\ &= 218 \end{aligned}$$

   c) Which term has a value of 322?
   Solution:

$$\begin{aligned} T_n &= 4n - 2 \\ \therefore 322 &= 4n - 2 \\ 324 &= 4n \\ \therefore n &= 81 \\ \therefore T_{81} &= 322 \end{aligned}$$

   d) Determine by calculation whether or not 1204 is a term in the sequence?
   Solution:

$$\begin{aligned} T_n &= 4n - 2 \\ \therefore 1204 &= 4n - 2 \\ 1206 &= 4n \\ \therefore n &= 301\frac{1}{2} \end{aligned}$$

   This value of n is not a positive integer, therefore 1204 is not a term of this sequence.

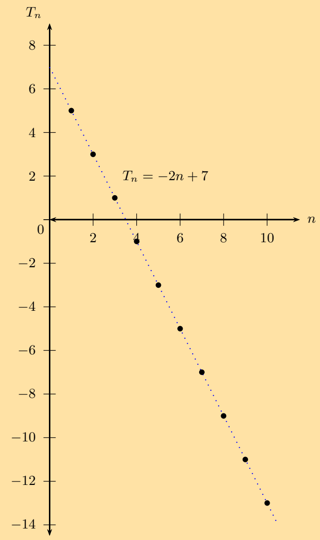
3. An arithmetic sequence has the general term $T_n = -2n + 7$.

   a) Calculate the second, third and tenth terms of the sequence.
   Solution:

$$\begin{aligned} T_2 &= -2(2) + 7 = 3 \\ T_3 &= -2(3) + 7 = 1 \\ T_{10} &= -2(10) + 7 = -13 \end{aligned}$$

b) Draw a diagram of the sequence for $0 < n \leq 10$.

Solution:



4. The first term of an arithmetic sequence is $-\frac{1}{2}$ and $T_{22} = 10$. Find T_n.

Solution:

Calculate the common difference (d):

$$
\begin{aligned}
a &= -\frac{1}{2} \\
\text{General formula: } T_n &= a + (n-1)d \\
T_{22} &= -\frac{1}{2} + (22-1)d \\
\therefore 10 &= -\frac{1}{2} + 21d \\
10 + \frac{1}{2} &= 21d \\
\frac{21}{2} &= 21d \\
\therefore d &= \frac{1}{2}
\end{aligned}
$$

Now determine the general term for the sequence:

$$
\begin{aligned}
T_n &= a + (n-1)d \\
T_n &= -\frac{1}{2} + (n-1)\left(\frac{1}{2}\right) \\
&= -\frac{1}{2} + \frac{1}{2}n - \frac{1}{2} \\
\therefore T_n &= \frac{1}{2}n - 1
\end{aligned}
$$

5. What are the important characteristics of an arithmetic sequence?

Solution:

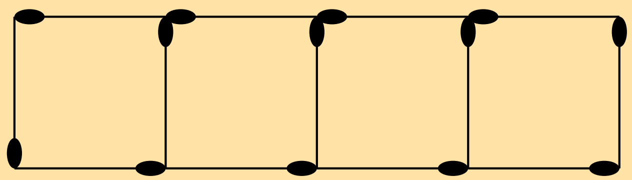
- There is a common difference between any two successive terms in the sequence.
- The graph of T_n vs. n is a straight line.

6. You are given the first four terms of an arithmetic sequence. Describe the method you would use to find the formula for the n^{th} term of the sequence.

Solution:

- Use the given terms to calculate the common difference (d) : $\quad d = T_2 - T_1$.
- From given terms, we know that $T_1 = a$.
- Substitute the values for a and d into the equation $T_n = a + (n-1)d$.
- Simplify and gather like n terms.

7. A single square is made from 4 matchsticks. To make two squares in a row takes 7 matchsticks, while three squares in a row takes 10 matchsticks.



a) Write down the first four terms of the sequence.

Solution: $4; 7; 10; 13$

b) What is the common difference?

Solution:

$$\begin{aligned} d &= T_2 - T_1 \\ &= 7 - 4 \\ &= 3 \end{aligned}$$

c) Determine the formula for the general term.

Solution:

$$\begin{aligned} a &= 4 \\ d &= 3 \\ T_n &= a + (n-1)d \\ &= 4 + (n-1)(3) \\ &= 4 + 3n - 3 \\ \therefore T_n &= 3n + 1 \end{aligned}$$

d) How many matchsticks are in a row of 25 squares?

Solution:

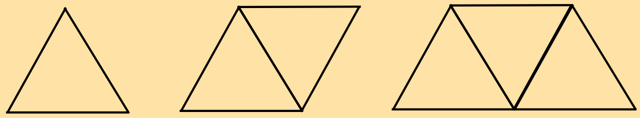
$$\begin{aligned} T_n &= a + (n-1)d \\ T_{25} &= 4 + (25-1)(3) \\ &= 4 + (24)(3) \\ &= 76 \end{aligned}$$

e) If there are 109 matchsticks, calculate the number of squares in the row.

Solution:

$$\begin{aligned} T_n &= 3n + 1 \\ 109 &= 3n + 1 \\ 108 &= 3n \\ \therefore n &= 36 \\ \therefore T_{36} &= 109 \end{aligned}$$

8. A pattern of equilateral triangles decorates the border of a girl's skirt. Each triangle is made by three stitches, each having a length of 1 cm.



a) Complete the table:

Figure no.	1	2	3	q	r	n
No. of stitches	3	5	p	15	71	s

Solution:

$$\begin{aligned} p &= 7 \\ q &= 7 \\ r &= 35 \\ s &= 2n + 1 \end{aligned}$$

b) The border of the skirt is 2 m in length. If the entire length of the border is decorated with the triangular pattern, how many stitches will there be?

Solution:

$$\begin{aligned} 2\text{ m} &= 200\text{ cm} \\ 2\text{ triangles } &= 1\text{ cm in the border} \\ \therefore 2 \times 200 &= 400\text{ triangles in the border} \\ \text{No. of stitches: } T_n &= 2n + 1 \\ &= 2(400) + 1 \\ &= 801 \end{aligned}$$

9. The terms $p; (2p + 2); (5p + 3)$ form an arithmetic sequence. Find p and the 15^{e} term of the sequence.

[IEB, Nov 2011]

Solution:

$$\begin{aligned} d &= T_2 - T_1 \\ &= (2p + 2) - (p) \\ &= p + 2 \\ \text{or} & \\ d &= T_3 - T_2 \\ &= (5p + 3) - (2p + 2) \\ &= 3p + 1 \\ \therefore 3p + 1 &= p + 2 \\ 2p &= 1 \\ \therefore p &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} T_{15} &= a + 14d \\ &= p + 14(p + 2) \\ &= 15p + 28 \\ &= 15\left(\frac{1}{2}\right) + 28 \\ &= 35\frac{1}{2} \end{aligned}$$

10. The arithmetic mean of $3a - 2$ and x is $4a - 4$. Determine the value of x in terms of a.

Solution:

$$\begin{aligned}\frac{3a-2+x}{2} &= 4a-4\\ 3a-2+x &= 8a-8\\ \therefore x &= 5a-6\end{aligned}$$

or

$$\begin{aligned}T_2 - T_1 &= T_3 - T_2\\ \therefore (4a-4)-(3a-2) &= (x)-(4a-4)\\ 4a-4-3a+2 &= x-4a+4\\ \therefore 5a-6 &= x\end{aligned}$$

11. Insert seven arithmetic means between the terms $(3s-t)$ and $(-13s+7t)$.

Solution:
Let the arithmetic means be: $a_1; a_2; a_3; a_4; a_5; a_6; a_7$
Therefore the sequence is:

$$T_1; a_1; a_2; a_3; a_4; a_5; a_6; a_7; T_9$$

$$\begin{aligned}T_1 &= 3s-t\\ T_9 &= -13s+7t\\ d &= \frac{T_9-T_1}{8}\\ &= \frac{-13s+7t-3s+t}{8}\\ &= \frac{-16s+8t}{8}\\ &= -2s+t\\ \therefore T_1 &= 3s-t\\ a_1 &= (3s-t)+(-2s+t)\\ &= s\\ a_2 &= (s)+(-2s+t)\\ &= -s+t\\ a_3 &= (-s+t)+(-2s+t)\\ &= -3s+2t\\ a_4 &= (-3s+2t)+(-2s+t)\\ &= -5s+3t\\ a_5 &= (-5s+3t)+(-2s+t)\\ &= -7s+4t\\ a_6 &= (-7s+4t)+(-2s+t)\\ &= -9s+5t\\ a_7 &= (-9s+5t)+(-2s+t)\\ &= -11s+6t\\ T_9 &= (-11s+6t)+(-2s+t)\\ &= -13s+7t\end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 284Q 2. 284R 3. 284S 4. 284T 5. 284V 6. 284W
7. 284X 8. 284Y 9. 284Z 10. 2852 11. 2853

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Exercise 1 – 3: Quadratic sequences

1. Determine whether each of the following sequences is:

   - a linear sequence;
   - a quadratic sequence;
   - or neither.

   a) $8; 17; 32; 53; 80; \ldots$

   Solution:

   $$\begin{aligned}\text{First differences: } &= 9; 15; 21; 27\\ \text{Second difference: } &= 6\end{aligned}$$

   Quadratic sequence

   b) $3p^2; 6p^2; 9p^2; 12p^2; 15p^2; \ldots$

   Solution:

   $$\text{First difference: } = 3p^2$$

   Linear sequence

   c) $1; 2{,}5; 5; 8{,}5; 13; \ldots$

   Solution:

   $$\begin{aligned}\text{First differences: } &= 1{,}5;\ 2{,}5;\ 3{,}5;\ 4{,}5\\ \text{Second difference: } &= 1\end{aligned}$$

   Quadratic sequence

   d) $2; 6; 10; 14; 18; \ldots$

   Solution:

   $$\text{First difference: } = 4$$

   Linear sequence

   e) $5; 19; 41; 71; 109; \ldots$

   Solution:

   $$\begin{aligned}\text{First differences: } &= 14; 22; 30; 38\\ \text{Second difference: } &= 8\end{aligned}$$

   Quadratic sequence

   f) $3; 9; 16; 21; 27; \ldots$

   Solution:

   Neither

   g) $2k; 8k; 18k; 32k; 50k; \ldots$

   Solution:

   $$\begin{aligned}\text{First differences: } &= 6k; 10k; 14k; 18k\\ \text{Second difference: } &= 4k\end{aligned}$$

   Quadratic sequence

   h) $2\frac{1}{2}; 6; 10\frac{1}{2}; 16; 22\frac{1}{2}; \ldots$

   Solution:

   $$\begin{aligned}\text{First differences: } &= 3{,}5;\ 4{,}5;\ 5{,}5;\ 6{,}5\\ \text{Second difference: } &= 1\end{aligned}$$

   Quadratic sequence

2. A quadratic pattern is given by $T_n = n^2 + bn + c$. Find the values of b and c if the sequence starts with the following terms:

$$-1 \; ; \; 2 \; ; \; 7 \; ; \; 14 \; ; \; \ldots$$

Solution:

Starting with the first term, we have $n = 1$ and $T_1 = -1$:

$$\begin{aligned} T_1 &= (1)^2 + b(1) + c \\ (-1) &= 1 + b + c \\ -2 &= b + c \end{aligned}$$

For the second term, we use $n = 2$ and $T_2 = 2$:

$$\begin{aligned} T_2 &= (2)^2 + b(2) + c \\ (2) &= 4 + 2b + c \\ -2 &= 2b + c \end{aligned}$$

Now we must solve these equations simultaneously. We can do this by substitution, but here we will show the solution using the 'elmination' method (which means subtracting one equation from the other to cancel the c's).

$$\begin{aligned} -2 &= 2b + c \\ -(-2 &= \; b + c) \\ 0 &= \; b \end{aligned}$$

Finally, calculate the value of c. As usual for simultaneous equations, this means that we must substitute the $b = 0$ into either of the equations we used above. Let's use the equation $-2 = b + c$.

$$\begin{aligned} b = 0 \;\longrightarrow \quad -2 &= b + c \\ -2 &= (0) + c \\ -2 &= \; c \end{aligned}$$

The final answers are $b = 0$ and $c = -2$.

NOTE: Now we know that the general term of the sequence is $T_n = n^2 - 2$. We can use this to check our answers. We know that $T_3 = 7$. Substitute $n = 3$ into the general formula to check:

$$\begin{aligned} T_n &= n^2 - 2 \\ T_3 &= (3)^2 - 2 \\ &= (9) + 0 - 2 \\ &= 7 \end{aligned}$$

3. $a^2; -a^2; -3a^2; -5a^2; \ldots$ are the first 4 terms of a sequence.

   a) Is the sequence linear or quadratic? Motivate your answer.

   Solution:

$$\begin{aligned} T_2 - T_1 &= -a^2 - a^2 = -2a^2 \\ T_3 - T_2 &= -3a^2 - (-a^2) = -2a^2 \\ T_4 - T_3 &= -5a^2 - (-3a^2) = -2a^2 \end{aligned}$$

   This is an arithmetic sequence since there is a common difference of $-2a^2$ between consecutive terms.

b) What is the next term in the sequence?

Solution:

$$\begin{aligned} T_5 &= -5a^2 + \left(-2a^2\right) \\ &= -7a^2 \end{aligned}$$

c) Calculate T_{100}.

Solution:

$$\begin{aligned} T_n &= a + (n-1)d \\ \therefore T_{100} &= a^2 + (99)(-2a^2) \\ &= a^2 - 198a^2 \\ \therefore T_{100} &= -197a^2 \end{aligned}$$

4. Given $T_n = n^2 + bn + c$, determine the values of b and c if the sequence starts with the terms:

$$2 \; ; \; 7 \; ; \; 14 \; ; \; 23 \; ; \; \ldots$$

Solution:

Starting with the first term, we have $n = 1$ and $T_1 = 2$:

$$\begin{aligned} T_1 &= (1)^2 + b(1) + c \\ (2) &= 1 + b + c \\ 1 &= b + c \end{aligned}$$

For the second term, we use $n = 2$ and $T_2 = 7$:

$$\begin{aligned} T_2 &= (2)^2 + b(2) + c \\ (7) &= 4 + 2b + c \\ 3 &= 2b + c \end{aligned}$$

Now we must solve these equations simultaneously. We can do this by substitution, but here we will show the solution using the 'elimination' method (which means subtracting one equation from the other to cancel the c's).

$$\begin{aligned} 3 &= 2b + c \\ -(1 &= \;\; b + c) \\ 2 &= \;\; b \end{aligned}$$

Finally, calculate the value of c. As usual for simultaneous equations, this means that we must substitute the $b = 2$ into either of the equations we used above. Let's use the equation $1 = b + c$.

$$\begin{aligned} b = 2 \;\longrightarrow\quad 1 &= b + c \\ 1 &= (2) + c \\ -1 &= \;\; c \end{aligned}$$

The final answers are $b = 2$ and $c = -1$.

NOTE: Now we know that the general term of the sequence is $T_n = n^2 + 2n - 1$. We can use this to check our answers. We know that $T_3 = 14$. Substitute $n = 3$ into the general formula to check:

$$\begin{aligned} T_n &= n^2 + 2n - 1 \\ T_3 &= (3)^2 + 2(3) - 1 \\ &= (9) + 6 - 1 \\ &= 14 \end{aligned}$$

5. The first term of a quadratic sequence is 4, the third term is 34 and the common second difference is 10. Determine the first six terms in the sequence.

Solution:

$$\begin{aligned}
\text{Let } T_2 &= x \\
\therefore T_2 - T_1 &= x - 4 \\
\text{And } T_3 - T_2 &= 34 - x \\
\text{Second difference } &= (T_3 - T_2) - (T_2 - T_1) \\
&= (34 - x) - (x - 4) \\
\therefore 10 &= 38 - 2x \\
2x &= 28 \\
\therefore x &= 14
\end{aligned}$$

$4; 14; 34; 64; 104; 154$

6. A quadratic sequence has a second term equal to 1, a third term equal to -6 and a fourth term equal to -14.

   a) Determine the second difference for this sequence.

   Solution:

$$\begin{aligned}
T_3 - T_2 &= -6 - (1) \\
&= -7 \\
T_4 - T_3 &= -14 - (-6) \\
&= -8 \\
\therefore \text{Second difference} &= -1
\end{aligned}$$

   b) Hence, or otherwise, calculate the first term of the pattern.

   Solution:

$$\begin{aligned}
T_1 &= 1 + 6 \\
&= 7
\end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 2854 1b. 2855 1c. 2856 1d. 2857 1e. 2858 1f. 2859
1g. 285B 1h. 285C 2. 285D 3. 285F 4. 285G 5. 285H
6. 285J

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1.2 Geometric sequences

Exercise 1 – 4: Constant ratio of a geometric sequence

Determine the constant ratios for the following geometric sequences and write down the next three terms in each sequence:

1. $5; 10; 20; \ldots$

   Solution:

$$r = \frac{T_2}{T_1} = \frac{10}{5}$$
$$= 2$$
$$\therefore \text{Next terms: } 40; 80; 160$$

2. $\frac{1}{2}; \frac{1}{4}; \frac{1}{8}; \ldots$

   Solution:

$$r = \frac{T_2}{T_1} = \frac{\frac{1}{4}}{\frac{1}{2}}$$
$$= \frac{1}{2}$$
$$\therefore \text{Next terms: } \frac{1}{16}; \frac{1}{32}; \frac{1}{64}$$

3. $7; 0{,}7; 0{,}07; \ldots$

   Solution:

$$r = \frac{T_2}{T_1} = \frac{0{,}7}{7}$$
$$= 0{,}1$$
$$\therefore \text{Next terms: } 0{,}007; 0{,}0007; 0{,}00007$$

4. $p; 3p^2; 9p^3; \ldots$

   Solution:

$$r = \frac{T_2}{T_1} = \frac{3p^2}{p}$$
$$= 3p$$
$$\therefore \text{Next terms: } 27p^4; 81p^5; 243p^6$$

5. $-3; 30; -300; \ldots$

   Solution:

$$r = \frac{T_2}{T_1} = \frac{30}{-3}$$
$$= -10$$
$$\therefore \text{Next terms: } 3000; -30\,000; 300\,000; -3\,000\,000;$$

Check answers online with the exercise code below or click on 'show me the answer'.
1. 285M 2. 285N 3. 285P 4. 285Q 5. 285R

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Exercise 1 – 5: General term of a geometric sequence

Determine the general formula for the n^{th} term of each of the following geometric sequences:

1. $5; 10; 20; \ldots$

   Solution:

$$\begin{aligned} a &= 5 \\ r = \frac{T_2}{T_1} &= \frac{10}{5} \\ &= 2 \\ T_n &= ar^{n-1} \\ \therefore T_n &= 5(2)^{n-1} \end{aligned}$$

2. $\frac{1}{2}; \frac{1}{4}; \frac{1}{8}; \ldots$

   Solution:

$$\begin{aligned} a &= \frac{1}{2} \\ r = \frac{T_2}{T_1} &= \frac{\frac{1}{4}}{\frac{1}{2}} \\ &= \frac{1}{2} \\ T_n &= ar^{n-1} \\ \therefore T_n &= \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} \\ &= \left(\frac{1}{2}\right)^{n} \end{aligned}$$

3. $7; 0{,}7; 0{,}07; \ldots$

   Solution:

$$\begin{aligned} a &= 7 \\ r = \frac{T_2}{T_1} &= \frac{0{,}7}{7} \\ &= 0{,}1 \\ T_n &= ar^{n-1} \\ \therefore T_n &= 7(0{,}1)^{n-1} \end{aligned}$$

4. $p; 3p^2; 9p^3; \ldots$

   Solution:

$$\begin{aligned} a &= p \\ r = \frac{T_2}{T_1} &= \frac{3p^2}{p} \\ &= 3p \\ T_n &= ar^{n-1} \\ \therefore T_n &= p(3p)^{n-1} \end{aligned}$$

5. $-3; 30; -300; \ldots$

Solution:

$$\begin{aligned} a &= -3 \\ r = \frac{T_2}{T_1} &= \frac{30}{-3} \\ &= -10 \\ T_n &= ar^{n-1} \\ \therefore T_n &= -3(-10)^{n-1} \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.
1. 285S 2. 285T 3. 285V 4. 285W 5. 285X

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Exercise 1 – 6: Mixed exercises

1. The n^{th} term of a sequence is given by the formula $T_n = 6\left(\frac{1}{3}\right)^{n-1}$.

   a) Write down the first three terms of the sequence.

   Solution:

$$\begin{aligned} T_1 &= 6\left(\frac{1}{3}\right)^0 \\ &= 6 \\ T_2 &= 6\left(\frac{1}{3}\right)^1 \\ &= 2 \\ T_3 &= 6\left(\frac{1}{3}\right)^2 \\ &= \frac{2}{3} \end{aligned}$$

   $\therefore 6; 2; \frac{2}{3} \ldots$

   b) What type of sequence is this?

   Solution:

$$\begin{aligned} r &= \frac{T_2}{T_1} \\ &= \frac{2}{6} \\ &= \frac{1}{3} \\ \text{Check: } r &= \frac{T_3}{T_2} \\ &= \frac{\frac{2}{3}}{2} \\ &= \frac{1}{3} \end{aligned}$$

   Therefore this is a geometric sequence with constant ratio $r = \frac{1}{3}$.

2. Consider the following terms:

$$(k-4); (k+1); m; 5k$$

The first three terms form an arithmetic sequence and the last three terms form a geometric sequence. Determine the values of k and m if both are positive integers.

[IEB, Nov 2006]

Solution:

First consider the arithmetic sequence: $(k-4); (k+1); m$

$$\begin{aligned}
d &= T_2 - T_1 \\
&= (k+1) - (k-4) \\
&= k+1-k+4 \\
&= 5 \\
\text{And } d &= T_3 - T_2 \\
&= m - (k+1) \\
&= m-k-1 \\
\therefore 5 &= m-k-1 \\
k+6 &= m \ldots\ldots (1)
\end{aligned}$$

Now consider the geometric sequence: $(k+1); m; 5k$

$$\begin{aligned}
r &= \frac{T_2}{T_1} \\
&= \frac{m}{k+1} \\
\text{And } r &= \frac{T_3}{T_2} \\
&= \frac{5k}{m} \\
\therefore \frac{m}{k+1} &= \frac{5k}{m} \\
m^2 &= 5k(k+1) \\
m^2 &= 5k^2+5k \ldots\ldots (2) \\
\text{Substitute eqn } (1) \rightarrow (2): (k+6)^2 &= 5k^2+5k \\
k^2+12k+36 &= 5k^2+5k \\
4k^2-7k-36 &= 0 \\
(4k+9)(k-4) &= 0 \\
\therefore k &= -\frac{9}{4} \text{ or } k = 4 \\
\text{But } k &\in \mathbb{Z} \\
\text{Therefore } k &= 4 \\
\text{And } m &= k+6 \\
&= 4+6 \\
&= 10
\end{aligned}$$

Therefore $k = 4$ and $m = 10$ giving the terms $0; 5; 10; 20$

3. Given a geometric sequence with second term $\frac{1}{2}$ and ninth term 64.

   a) Determine the value of r.

   Solution:

$$T_2 = \frac{1}{2}$$
$$\therefore ar = \frac{1}{2}$$
$$T_9 = 64$$
$$\therefore ar^8 = 64$$
$$\frac{ar^8}{ar} = \frac{64}{\frac{1}{2}}$$
$$\therefore r^7 = 128$$
$$= 2^7$$
$$\therefore r = 2$$

b) Find the value of a.

Solution:

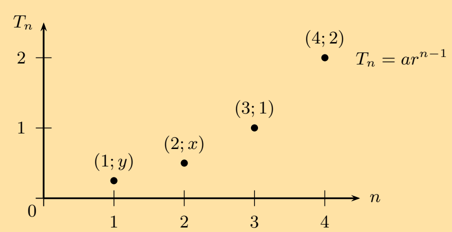
$$T_2 = \frac{1}{2}$$
$$\therefore ar = \frac{1}{2}$$
$$a(2) = \frac{1}{2}$$
$$\therefore a = \frac{1}{4}$$

c) Determine the general formula of the sequence.

Solution:

$$\begin{aligned} T_n &= ar^{n-1} \\ &= \frac{1}{4}(2)^{n-1} \\ &= 2^{-2} \, . \, 2^{n-1} \\ &= 2^{n-1-2} \\ &= 2^{n-3} \\ &= 2^n \, . \, 2^{-3} \\ &= \frac{2^n}{8} \end{aligned}$$

4. The diagram shows four sets of values of consecutive terms of a geometric sequence with the general formula $T_n = ar^{n-1}$.



a) Determine a and r.

Solution:

$$\begin{aligned}
(3;1): \quad T_3 &= ar^{3-1} \\
\therefore 1 &= ar^2 \\
\therefore \frac{1}{r^2} &= a \ldots\ldots (1) \\
(4;2): \quad T_4 &= ar^{4-1} \\
\therefore 2 &= ar^3 \ldots\ldots (2) \\
\text{Substitute } (1) \rightarrow (2): \quad 2 &= \left(\frac{1}{r^2}\right) r^3 \\
\therefore 2 &= r \\
\text{Substitute back into } (1): \quad a &= \frac{1}{(2)^2} \\
\therefore a &= \frac{1}{4} \\
\therefore T_n &= \frac{1}{4}(2)^{n-1}
\end{aligned}$$

b) Find x and y.

Solution:

Determine y:

$$\begin{aligned}
T_n &= \frac{1}{4}(2)^{n-1} \\
T_1 &= \frac{1}{4}(2)^{1-1} \\
\therefore y &= \frac{1}{4}
\end{aligned}$$

Determine x:

$$\begin{aligned}
T_n &= \frac{1}{4}(2)^{n-1} \\
T_2 &= \frac{1}{4}(2)^{2-1} \\
\therefore x &= \frac{1}{2}
\end{aligned}$$

c) Find the fifth term of the sequence.

Solution:

$$\begin{aligned}
T_n &= \frac{1}{4}(2)^{n-1} \\
T_5 &= \frac{1}{4}(2)^{5-1} \\
&= \frac{1}{4}(2)^4 \\
&= \frac{1}{4}(16) \\
&= 4
\end{aligned}$$

5. Write down the next two terms for the following sequence:

$$1; \sin\theta; 1-\cos^2\theta; \ldots$$

Solution:

Check if this is a geometric sequence:

$$\begin{aligned} r &= \frac{T_2}{T_1} \\ &= \sin\theta \\ \text{And } r &= \frac{T_3}{T_2} \\ &= \frac{1-\cos^2\theta}{\sin\theta} \\ &= \frac{\sin^2\theta}{\sin\theta} \\ &= \sin\theta \end{aligned}$$

This is a geometric sequence with $r = \sin\theta$. Therefore, $T_4 = \sin^3\theta$ and $T_5 = \sin^4\theta$.

6. $5; x; y$ is an arithmetic sequence and $x; y; 81$ is a geometric sequence. All terms in the sequences are integers. Calculate the values of x and y.

   Solution:

   For the arithmetic sequence:

$$\begin{aligned} d &= T_2 - T_1 \\ &= x - 5 \\ \text{And } d &= T_3 - T_2 \\ &= y - x \\ \therefore x - 5 &= y - x \\ 2x - 5 &= y \ldots\ldots (1) \end{aligned}$$

   For the geometric sequence:

$$\begin{aligned} r &= \frac{T_2}{T_1} \\ &= \frac{y}{x} \\ \text{And } r &= \frac{T_3}{T_2} \\ &= \frac{81}{y} \\ \therefore \frac{y}{x} &= \frac{81}{y} \\ y^2 &= 81x \ldots\ldots (2) \end{aligned}$$

$$\begin{aligned} \text{Substitute eqn } (1) \rightarrow (2): \quad (2x-5)^2 &= 81x \\ 4x^2 - 20x + 25 &= 81x \\ 4x^2 - 101x + 25 &= 0 \\ (4x-1)(x-25) &= 0 \\ \therefore x &= \frac{1}{4} \text{ or } x = 25 \end{aligned}$$

$$\begin{aligned} \text{If } x = \frac{1}{4}: \quad y^2 &= 81x \\ &= \frac{81}{4} \\ \therefore y &= \pm\frac{9}{2} \\ \text{If } x = 25: y^2 &= 81x \\ &= 81 \times 25 \\ &= 2025 \\ \therefore y &= \pm 45 \end{aligned}$$

$$\text{Arithmetic sequence: } 5; \frac{1}{4}; -\frac{9}{2}; \ldots$$
$$\text{or} \quad 5; 25; 45; \ldots$$
$$\text{Geometric sequence: } \frac{1}{4}; -\frac{9}{2}; 81; \ldots$$
$$\text{or} \quad 25; -45; 81; \ldots$$

7. The two numbers $2x^2y^2$ and $8x^4$ are given.

   a) Write down the geometric mean between the two numbers in terms of x and y.
   Solution:

$$\begin{aligned}
\text{Let the mean be } T_2 &= p \\
\frac{T_2}{T_1} &= \frac{T_3}{T_2} \\
\frac{p}{2x^2y^2} &= \frac{8x^4}{p} \\
\therefore p^2 &= (2x^2y^2)(8x^4) \\
p^2 &= 16x^6y^2 \\
\therefore p &= 4x^3y
\end{aligned}$$

   Note: in this case only the positive square root is valid.

   b) Determine the constant ratio of the resulting sequence.
   Solution:

$$\begin{aligned}
r &= \frac{T_2}{T_1} \\
&= \frac{4x^3y}{2x^2y^2} \\
&= \frac{2x}{y}
\end{aligned}$$

8. Insert three geometric means between -1 and $-\frac{1}{81}$. Give all possible answers.

   Solution:
   Let the geometric sequence be $-1; T_2; T_3; T_4; -\frac{1}{81}$

$$\begin{aligned}
T_1 &= -1 = a \\
T_5 &= -\frac{1}{81} = ar^4 \\
\therefore (-1)r^4 &= -\frac{1}{81} \\
r^4 &= \frac{1}{81} \\
\therefore r &= \pm\frac{1}{3}
\end{aligned}$$

   Therefore possible geometric sequences are:
   $-1; \frac{1}{3}; -\frac{1}{9}; \frac{1}{27}; -\frac{1}{81}$
   $-1; -\frac{1}{3}; -\frac{1}{9}; -\frac{1}{27}; -\frac{1}{81}$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 285Y 2. 285Z 3. 2862 4. 2863 5. 2864 6. 2865
7. 2866 8. 2867

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1.3 Series

Sigma notation

Exercise 1 – 7: Sigma notation

1. Determine the value of the following:

a)

$$\sum_{k=1}^{4} 2$$

Solution:

$2 + 2 + 2 + 2 = 8$

b)

$$\sum_{i=-1}^{3} i$$

Solution:

$$\begin{aligned}\sum_{i=-1}^{3} i &= -1 + 0 + 1 + 2 + 3 \\ &= 5\end{aligned}$$

c)

$$\sum_{n=2}^{5} (3n - 2)$$

Solution:

$$\begin{aligned}\sum_{n=2}^{5} (3n - 2) &= [3(2) - 2] + [3(3) - 2] + [3(4) - 2] + [3(5) - 2] \\ &= 4 + 7 + 10 + 13 \\ &= 34\end{aligned}$$

2. Expand the series:

a)

$$\sum_{k=1}^{6} 0^k$$

Solution:

$0^1 + 0^2 + 0^3 + 0^4 + 0^5 + 0^6 = 0$

b)

$$\sum_{n=-3}^{0} 8$$

Solution:

$8 + 8 + 8 + 8 = 32$

c)

$$\sum_{k=1}^{5} (ak)$$

Solution:

$$\begin{aligned}\sum_{k=1}^{5}(ak) &= a + 2a + 3a + 4a + 5a \\ &= 15a\end{aligned}$$

3. Calculate the value of a:

a)

$$\sum_{k=1}^{3}\left(a\ .\ 2^{k-1}\right) = 28$$

Solution:

$$\begin{aligned}\sum_{k=1}^{3}\left(a\ .\ 2^{k-1}\right) &= 28 \\ \therefore a + 2a + 4a &= 28 \\ 7a &= 28 \\ \therefore a &= 4\end{aligned}$$

b)

$$\sum_{j=1}^{4}\left(2^{-j}\right) = a$$

Solution:

$$\begin{aligned}\sum_{j=1}^{4}\left(2^{-j}\right) &= a \\ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} &= a \\ \therefore \frac{15}{16} &= a\end{aligned}$$

4. Write the following in sigma notation:

$$\frac{1}{9} + \frac{1}{3} + 1 + 3$$

Solution:

Geometric series with $a = \frac{1}{9}$, constant ratio $r = 3$ and general formula $T_n = ar^{n-1}$.

$$\begin{aligned}T_n &= ar^{n-1} \\ &= \frac{1}{9}(3)^{n-1} \\ &= 3^{-2}\ .\ 3^{n-1} \\ &= 3^{n-3} \\ \therefore &\sum_{n=1}^{4}\left(3^{n-3}\right)\end{aligned}$$

5. Write the sum of the first 25 terms of the series below in sigma notation:

$$11 + 4 - 3 - 10 \ldots$$

Solution:

Arithmetic series with $a = 11$, common difference $d = -7$ and general formula $T_n = a + (n-1)d$.

$$\begin{aligned} T_n &= a + (n-1)d \\ &= 11 + (n-1)(-7) \\ &= 11 - 7n + 7 \\ &= 18 - 7n \\ \therefore &\sum_{n=1}^{25} (18 - 7n) \end{aligned}$$

6. Write the sum of the first 1000 natural, odd numbers in sigma notation.

Solution:

$$1 + 3 + 5 + 7 + \ldots$$

Arithmetic series with $a = 1$, common difference $d = 2$ and general formula $T_n = a + (n-1)d$.

$$\begin{aligned} T_n &= a + (n-1)d \\ &= 1 + (n-1)(2) \\ &= 1 + 2n - 2 \\ &= 2n - 1 \\ \therefore &\sum_{n=1}^{1000} (2n - 1) \\ \text{or } &\sum_{n=0}^{999} (2n + 1) \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 2868 1b. 2869 1c. 286B 2a. 286C 2b. 286D 2c. 286F
3a. 286G 3b. 286H 4. 286J 5. 286K 6. 286M

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1.4 Finite arithmetic series

General formula for a finite arithmetic series

Exercise 1 – 8: Sum of an arithmetic series

1. Determine the value of k:

$$\sum_{n=1}^{k} (-2n) = -20$$

Solution:

$$
\begin{aligned}
(-2(1)) + (-2(2)) + (-2(3)) + \ldots + (-2(k)) &= -20 \\
-2 - 4 - 6 + \ldots - 2k &= -20
\end{aligned}
$$

This is an arithmetic series with $a = -2$ and $d = -2$:

$$
\begin{aligned}
S_n &= \frac{n}{2}[2a + (n-1)d] \\
-20 &= \frac{n}{2}[2(-2) + (n-1)(-2)] \\
-40 &= n[-4 + -2n + 2] \\
-40 &= n[-2n - 2] \\
-40 &= -2n^2 - 2n \\
2n^2 + 2n - 40 &= 0 \\
n^2 + n - 20 &= 0 \\
(n+5)(n-4) &= 0 \\
\therefore n &= -5 \text{ or } n = 4 \\
\therefore S_4 &= -20 \\
\therefore k &= 4
\end{aligned}
$$

2. The sum to n terms of an arithmetic series is $S_n = \frac{n}{2}(7n + 15)$.

   a) How many terms of the series must be added to give a sum of 425?

   Solution:

$$
\begin{aligned}
S_n &= \frac{n}{2}(7n + 15) \\
\therefore 425 &= \frac{n}{2}(7n + 15) \\
850 &= n(7n + 15) \\
&= 7n^2 + 15n \\
0 &= 7n^2 + 15n - 850 \\
&= (7n + 85)(n - 10) \\
\therefore n &= -\frac{85}{7} \text{ or } n = 10
\end{aligned}
$$

   but n must be a positive integer, therefore $n = 10$.

   b) Determine the sixth term of the series.

   Solution:

$$
\begin{aligned}
S_n &= \frac{n}{2}(7n+15) \\
S_1 &= T_1 = a \\
S_1 &= \frac{n}{2}(7n+15) \\
&= \frac{1}{2}(7(1)+15) \\
\therefore a &= 11 \\
S_2 &= \frac{2}{2}(7(2)+15) \\
&= 29 \\
\therefore T_1 + T_2 &= 29 \\
\therefore T_2 &= 29 - 11 \\
\text{And } d &= T_2 - T_1 \\
&= 18 - 11 \\
&= 7 \\
\therefore T_n &= a + (n-1)d \\
&= 11 + (n-1)(7) \\
&= 11 + 7n - 7 \\
&= 7n + 4 \\
\therefore T_6 &= 7(6) + 4 \\
&= 46
\end{aligned}
$$

3. a) The common difference of an arithmetic series is 3. Calculate the values of n for which the n^{th} term of the series is 93, and the sum of the first n terms is 975.

Solution:

$$
\begin{aligned}
d &= 3 \\
T_n &= a + (n-1)d \\
93 &= a + 3(n-1) \\
&= a + 3n - 3 \\
96 &= a + 3n \\
\therefore a &= 96 - 3n \\
S_n &= \frac{n}{2}[2a + (n-1)d] \\
\therefore 975 &= \frac{n}{2}[2(96-3n) + 3(n-1)] \\
1950 &= n[192 - 6n + 3n - 3] \\
1950 &= 189n - 3n^2 \\
0 &= -3n^2 + 189n - 1950 \\
0 &= n^2 - 63n + 650 \\
0 &= (n-13)(n-50) \\
\therefore n &= 13 \text{ or } n = 50
\end{aligned}
$$

b) Explain why there are two possible answers.

Solution:

There are two series that satisfy the given parameters:

$$d = 3$$
$$a = 96 - 3n$$
$$\text{If } n = 13$$
$$a = 96 - 3(13)$$
$$= 57$$
$$\therefore 57 + 60 + 63 + \ldots + T_{13} = 975$$
$$\text{If } n = 50$$
$$a = 96 - 3(50)$$
$$= -54$$
$$\therefore (-54) + (-51) + (-48) + \ldots + T_{50} = 975$$

4. The third term of an arithmetic sequence is -7 and the seventh term is 9. Determine the sum of the first 51 terms of the sequence.

   Solution:

$$T_3 = -7 = a + 2d \ldots\ldots (1)$$
$$T_7 = 9 = a + 6d \ldots\ldots (2)$$
$$\therefore \text{Subtract eqns: } (1) - (2) \quad -7 - (9) = a + 2d - (a + 6d)$$
$$-16 = -4d$$
$$\therefore 4 = d$$
$$\text{Substitute back into eqn. } (1) \quad a = -7 - 2(4)$$
$$\therefore a = -15$$
$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$S_{51} = \frac{51}{2}[2(-15) + (51-1)(4)]$$
$$= \frac{51}{2}[-30 + 200]$$
$$= (51)(85)$$
$$\therefore S_{51} = 4335$$

5. Calculate the sum of the arithmetic series $4 + 7 + 10 + \cdots + 901$.

   Solution:

$$a = 4$$
$$l = 901$$
$$d = T_2 - T_1$$
$$= 7 - 4$$
$$= 3$$
$$\text{And } T_n = a + (n-1)d$$
$$= 4 + (n-1)(3)$$
$$\therefore 901 = 4 + 3n - 3$$
$$900 = 3n$$
$$\therefore 300 = n$$
$$S_n = \frac{n}{2}[a + l]$$
$$S_{300} = \frac{300}{2}[4 + 901]$$
$$= (150)(905)$$
$$\therefore S_{300} = 135750$$

6. Evaluate without using a calculator: $\dfrac{4 + 8 + 12 + \cdots + 100}{3 + 10 + 17 + \cdots + 101}$

Solution:

Consider the numerator: $4 + 8 + 12 + \ldots + 100$

$$\begin{aligned}
a &= 4 \\
l &= 100 \\
d &= T_2 - T_1 \\
&= 8 - 4 \\
&= 4 \\
\text{And } T_n &= a + (n-1)d \\
100 &= 4 + (n-1)(4) \\
100 &= 4n \\
\therefore 25 &= n \\
S_n &= \frac{n}{2}[a + l] \\
S_{25} &= \frac{25}{2}[4 + 100] \\
S_{25} &= (25)(52)
\end{aligned}$$

Consider the denominator: $3 + 10 + 17 + \ldots + 101$

$$\begin{aligned}
a &= 3 \\
l &= 101 \\
d &= T_2 - T_1 \\
&= 10 - 3 \\
&= 7 \\
\text{And } T_n &= a + (n-1)d \\
101 &= 3 + (n-1)(7) \\
101 &= 3 + 7n - 7 \\
105 &= 7n \\
\therefore 15 &= n \\
S_n &= \frac{n}{2}[a + l] \\
S_{15} &= \frac{15}{2}[3 + 101] \\
S_{15} &= (15)(52)
\end{aligned}$$

Now consider the quotient of the two series:

$$\begin{aligned}
\frac{S_{25}}{S_{15}} &= \frac{25 \times 52}{15 \times 52} \\
&= \frac{25}{15} \\
&= \frac{5}{3}
\end{aligned}$$

7. The second term of an arithmetic sequence is -4 and the sum of the first six terms of the series is 21.

   a) Find the first term and the common difference.

   Solution:

$$\begin{aligned}
T_n &= a + (n-1)d \\
T_2 &= a + d \\
-4 &= a + d \ldots\ldots (1) \\
S_n &= \frac{n}{2}[2a + (n-1)d] \\
S_6 &= \frac{6}{2}[2a + (6-1)d] \\
21 &= 3[2a + 5d] \\
\therefore 7 &= 2a + 5d \ldots\ldots (2) \\
\text{Eqn. } (1) \times 2: \quad -8 &= 2a + 2d \\
\text{Eqn. } (2) - 2(1): \quad 7 - (-8) &= (2a + 5d) - (2a + 2d) \\
15 &= 3d \\
\therefore 5 &= d \\
\text{And } a &= -4 - 5 \\
&= -9
\end{aligned}$$

b) Hence determine T_{100}.
[IEB, Nov 2004]

Solution:

$$\begin{aligned}
T_n &= a + (n-1)d \\
T_{100} &= -9 + (100-1)(5) \\
&= -9 + 495 \\
&= 486
\end{aligned}$$

8. Determine the value of the following:

a)

$$\sum_{w=0}^{8} (7w + 8)$$

Solution:
Arithmetic series: $8 + 15 + 22 + \ldots + 64$

$$\begin{aligned}
S_n &= \frac{n}{2}[2a + (n-1)d] \\
a &= 8 \\
d &= 15 - 8 = 7 \\
\therefore S_9 &= \frac{9}{2}[2(8) + (9-1)(7)] \\
&= \frac{9}{2}[16 + 56] \\
&= \frac{9}{2}[72] \\
&= (9)(36) \\
&= 324
\end{aligned}$$

b)

$$\sum_{j=1}^{8} 7j + 8$$

Solution:
Arithmetic series: $7 + 14 + 21 + \ldots + 56$

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ a &= 7 \\ d &= 14 - 7 = 7 \\ \therefore S_8 &= \frac{8}{2}[2(7) + (8-1)(7)] \\ &= 4[14 + 49] \\ &= 4(63) \\ &= 252 \\ \therefore S_8 + 8 &= 260 \end{aligned}$$

$$\begin{aligned} \text{Or } S_n &= \frac{n}{2}[a + l] \\ a &= 7 \\ l &= 56 \\ \therefore S_8 &= \frac{8}{2}[7 + 56] \\ &= 4(63) \\ &= 252 \\ \therefore S_8 + 8 &= 260 \end{aligned}$$

9. Determine the value of n.

$$\sum_{c=1}^{n} (2 - 3c) = -330$$

Solution:

Series: $-1 - 4 - 7 \ldots + (2 - 3n)$

$$\begin{aligned} a &= -1 \\ d &= T_2 - T_1 = -4 - (-1) = -3 \\ d &= T_3 - T_2 = -7 - (-4) = -3 \\ &\therefore \text{this is an arithmetic series} \\ \therefore S_n &= \frac{n}{2}[2a + (n-1)d] \\ -330 &= \frac{n}{2}[2(-1) + (n-1)(-3)] \\ -660 &= n[-2 - 3n + 3] \\ -660 &= n - 3n^2 \\ \therefore 0 &= -3n^2 + n + 660 \\ 0 &= 3n^2 - n - 660 \\ 0 &= (3n + 44)(n - 15) \\ \therefore n &= -\frac{44}{3} \text{ or } n = 15 \end{aligned}$$

but n must be a positive integer, therefore $n = 15$.

Alternative method:

$$\begin{aligned}
a &= -1 \\
l &= 2 - 3n \\
\therefore S_n &= \frac{n}{2}[a + 1] \\
-330 &= \frac{n}{2}[-1 + 2 - 3n] \\
-660 &= n(1 - 3n) \\
-660 &= n - 3n^2 \\
\therefore 0 &= -3n^2 + n + 660 \\
0 &= 3n^2 - n - 660 \\
0 &= (3n + 44)(n - 15) \\
\therefore n &= -\frac{44}{3} \text{ or } n = 15
\end{aligned}$$

but n must be a positive integer, therefore $n = 15$.

10. The sum of n terms of an arithmetic series is $5n^2 - 11n$ for all values of n. Determine the common difference.

Solution:

$$\begin{aligned}
S_n &= 5n^2 - 11n \\
\therefore S_1 &= 5(1)^2 - 11(1) \\
&= -6 \\
\text{And } S_2 &= 5(2)^2 - 11(2) \\
&= 20 - 22 \\
&= -2 \\
&= T_1 + T_2 \\
\therefore T_2 &= S_2 - S_1 \\
&= -2 - (-6) \\
&= 4 \\
\therefore d &= T_2 - T_1 \\
&= 4 - (-6) \\
&= 10
\end{aligned}$$

11. The sum of an arithmetic series is 100 times its first term, while the last term is 9 times the first term. Calculate the number of terms in the series if the first term is not equal to zero.

Solution:

$$\begin{aligned}
S_n &= 100a \\
l &= 9a \\
S_n &= \frac{n}{2}[a + l] \\
100a &= \frac{n}{2}[a + 9a] \\
100a &= \frac{n}{2}[10a] \\
100a &= 5a(n) \\
\frac{100a}{5a} &= n \\
\therefore 20 &= n
\end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 286P 2a. 286Q 2b. 286R 3. 286S 4. 286T 5. 286V
6. 286W 7. 286X 8a. 286Y 8b. 286Z 9. 2872 10. 2873
11. 2874

1.5 Finite geometric series

General formula for a finite geometric series

Exercise 1 – 9: Sum of a geometric series

1. Prove that $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n-1)}{r-1}$ and state any restrictions.

   Solution:

$$\begin{aligned} S_n &= a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \ldots (1) \\ r \times S_n &= \qquad ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n \ldots (2) \\ &\text{Subtract eqn. } (1) \text{ from eqn. } (2) \\ \therefore rS_n - S_n &= ar^n - a \\ S_n(r-1) &= a(r^n - 1) \\ \therefore S_n &= \frac{a(r^n-1)}{r-1} \end{aligned}$$

   where $r \neq 1$.

2. Given the geometric sequence $1; -3; 9; \ldots$ determine:

   a) The eighth term of the sequence.
   Solution:

$$\begin{aligned} a &= 1 \\ r &= \frac{T_2}{T_1} = -3 \\ T_n &= ar^{n-1} \\ \therefore T_8 &= (1)(-3)^{8-1} \\ &= (1)(-3)^7 \\ &= -2187 \end{aligned}$$

   b) The sum of the first eight terms of the sequence.
   Solution:

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} \\ \therefore S_8 &= \frac{(1)(1-(-3)^8)}{1-(-3)} \\ &= \frac{1-6561}{4} \\ &= -\frac{6560}{4} \\ &= -1640 \end{aligned}$$

3. Determine:

$$\sum_{n=1}^{4} 3 \,.\, 2^{n-1}$$

Solution:

$$\begin{aligned} S_4 &= 3 + 6 + 12 + 24 \\ &= 45 \end{aligned}$$

4. Find the sum of the first 11 terms of the geometric series $6 + 3 + \frac{3}{2} + \frac{3}{4} + \cdots$

Solution:

$$\begin{aligned} a &= 6 \\ r &= \frac{1}{2} \\ S_n &= \frac{a(1 - r^n)}{1 - r} \\ S_{11} &= \frac{6(1 - \left(\frac{1}{2}\right)^{11})}{1 - \left(\frac{1}{2}\right)} \\ &= 12\left(1 - \frac{1}{2048}\right) \\ &= 12\left(\frac{2047}{2048}\right) \\ &= \frac{6141}{512} \end{aligned}$$

5. Show that the sum of the first n terms of the geometric series $54 + 18 + 6 + \cdots + 5\left(\frac{1}{3}\right)^{n-1}$ is given by $\left(81 - 3^{4-n}\right)$.

Solution:

$$\begin{aligned} a &= 54 \\ r &= \frac{1}{3} \\ S_n &= \frac{a(1 - r^n)}{1 - r} \\ &= \frac{54(1 - \left(\frac{1}{3}\right)^n)}{1 - \left(\frac{1}{3}\right)} \\ &= \frac{54(1 - \left(\frac{1}{3}\right)^n)}{\frac{2}{3}} \\ &= 81(1 - 3^{-n}) \\ &= 81 - 81 \,.\, 3^{-n} \\ &= 81 - (3^4 \,.\, 3^{-n}) \\ &= 81 - 3^{4-n} \end{aligned}$$

6. The eighth term of a geometric sequence is 640. The third term is 20. Find the sum of the first 7 terms.

Solution:

$$T_8 = 640 = ar^7$$
$$T_3 = 20 = ar^2$$
$$\therefore \frac{T_8}{T_3} = \frac{640}{20}$$
$$\frac{640}{20} = \frac{ar^7}{ar^2}$$
$$32 = r^5$$
$$\therefore 2 = r$$
$$\text{And } 20 = ar^2$$
$$20 = a(2)^2$$
$$\frac{20}{4} = a$$
$$\therefore 5 = a$$
$$r = 2$$
$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_7 = \frac{5((2)^7 - 1)}{2 - 1}$$
$$= 5(128 - 1)$$
$$= 635$$

7. Given:

$$\sum_{t=1}^{n} 8\left(\frac{1}{2}\right)^t$$

a) Find the first three terms in the series.
Solution:

$$t = 1: \quad T_1 = 4$$
$$t = 2: \quad T_2 = 2$$
$$t = 3: \quad T_3 = 1$$
$$4;2;1$$

b) Calculate the number of terms in the series if $S_n = 7\frac{63}{64}$.
Solution:

$$a = 4$$
$$r = \frac{1}{2}$$
$$S_n = \frac{a(1 - r^n)}{1 - r}$$
$$\frac{511}{64} = \frac{4(1 - \left(\frac{1}{2}\right)^n)}{1 - \left(\frac{1}{2}\right)}$$
$$= \frac{4 - 4\left(\frac{1}{2}\right)^n}{\frac{1}{2}}$$
$$\frac{511}{128} = 4 - (2^2 \,.\, 2^{-n})$$
$$2^{2-n} = 4 - \frac{511}{128}$$
$$2^{2-n} = \frac{1}{128}$$
$$2^{2-n} = 2^{-7}$$
$$2 - n = -7$$
$$\therefore 9 = n$$

8. The ratio between the sum of the first three terms of a geometric series and the sum of the 4^{th}, 5^{th} and 6^{th} terms of the same series is $8 : 27$. Determine the constant ratio and the first 2 terms if the third term is 8.

Solution:

$$\begin{aligned}
T_1 + T_2 + T_3 &= a + ar + ar^2 \\
&= a(1 + r + r^2) \\
T_4 + T_5 + T_6 &= ar^3 + ar^4 + ar^5 \\
&= ar^3(1 + r + r^2) \\
\therefore \frac{T_1 + T_2 + T_3}{T_4 + T_5 + T_6} &= \frac{a(1 + r + r^2)}{ar^3(1 + r + r^2)} \\
\text{And} \quad \frac{T_1 + T_2 + T_3}{T_4 + T_5 + T_6} &= \frac{8}{27} \\
\therefore \frac{8}{27} &= \frac{a(1 + r + r^2)}{ar^3(1 + r + r^2)} \\
&= \frac{1}{r^3} \\
\therefore r^3 &= \frac{27}{8} \\
&= \left(\frac{3}{2}\right)^3 \\
\therefore r &= \frac{3}{2}
\end{aligned}$$

$$\begin{aligned}
\text{And } T_3 &= 8 \\
\therefore ar^2 &= 8 \\
a\left(\frac{3}{2}\right)^2 &= 8 \\
\therefore a &= 8 \times \frac{4}{9} \\
\therefore T_1 &= \frac{32}{9} \\
T_2 &= ar \\
&= \frac{32}{9} \times \frac{3}{2} \\
&= \frac{16}{3}
\end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 2876 2a. 2877 2b. 2878 3. 2879 4. 287B 5. 287C
6. 287D 7a. 287F 7b. 287G 8. 287H

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1.6 Infinite series

Exercise 1 – 10: Convergent and divergent series

For each of the general terms below:

- Determine if it forms an arithmetic or geometric series.
- Calculate S_1, S_2, S_{10} and S_{100}.
- Determine if the series is convergent or divergent.

1. $T_n = 2n$

   Solution:

   $2 + 4 + 6 + 8 + \ldots$

$$\begin{aligned} a &= 2 \\ d &= 2 \\ &\therefore \text{this is an arithmetic series} \\ S_1 &= 2 \\ S_2 &= 2 + 4 = 6 \\ S_n &= \frac{n}{2}(2a + [n-1]d) \\ S_{10} &= 5(2(2) + 9(2)) = 110 \\ S_{100} &= 50(2(2) + 99(2)) = 10100 \end{aligned}$$

$$\begin{aligned} a &= 2 \\ S_n &= \frac{n}{2}(2(2) + [n-1]2) \\ &= \frac{n}{2}(2 + 2n) \\ &= n(1+n) \\ &= n + n^2 \\ \therefore S_n &\to \infty \text{ as } n \to \infty \end{aligned}$$

   Therefore, this is a divergent series.

2. $T_n = (-n)$

   Solution:

   $(-1) + (-2) + (-3) + (-4) + \ldots$

$$\begin{aligned} a &= -1 \\ d &= -1 \\ &\therefore \text{this is an arithmetic series} \\ S_1 &= -1 \\ S_2 &= -1 - 2 = -3 \\ S_n &= \frac{n}{2}(2a + [n-1]d) \\ S_{10} &= 5(2(-1) + 9(-1)) = -55 \\ S_{100} &= 50(2(-1) + 99(-1)) = -5050 \end{aligned}$$

$$\begin{aligned} S_n &= \frac{n}{2}(2(-1) + [n-1](-1)) \\ &= \frac{n}{2}(-2 - n + 1) \\ &= -\frac{n}{2}(n+1) \\ \therefore S_n &\to -\infty \text{ as } n \to \infty \end{aligned}$$

   This is a divergent series.

3. $T_n = (\frac{2}{3})^n$

Solution:

$\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \ldots$

$$\begin{aligned}
a &= \frac{2}{3} \\
r &= \frac{2}{3} \\
&\therefore \text{this is a geometric series (with } r < 1) \\
S_1 &= \frac{2}{3} = 0{,}666\ldots \\
S_2 &= \frac{2}{3} + \frac{4}{9} = \frac{10}{9} = 1{,}111\ldots \\
S_n &= \frac{a(1-r^n)}{1-r} \\
S_{10} &= \frac{\frac{2}{3}\left(1-\left(\frac{2}{3}\right)^{10}\right)}{1-\frac{2}{3}} \\
&= 2\left(1-\left(\frac{2}{3}\right)^{10}\right) \\
&= 1{,}965\ldots \\
S_{100} &= \frac{\frac{2}{3}\left(1-\left(\frac{2}{3}\right)^{100}\right)}{1-\frac{2}{3}} \\
&= 2\left(1-\left(\frac{2}{3}\right)^{100}\right) \\
&= 2{,}00\ldots \\
S_n &= \frac{\frac{2}{3}(1-\left(\frac{2}{3}\right)^n)}{1-\frac{2}{3}} \\
&= \frac{\frac{2}{3}\left(1-\left(\frac{2}{3}\right)^n\right)}{\frac{1}{3}} \\
&= 2\left(1-\left(\frac{2}{3}\right)^n\right) \\
&= 2-2\left(\frac{2}{3}\right)^n
\end{aligned}$$

Therefore, as $n \to \infty \quad S_n \to 2$

This series is convergent (since the $r < 1$) and converges to 2.

4. $T_n = 2^n$

Solution:

$2 + 4 + 8 + 16 + \ldots$

$$
\begin{aligned}
a &= 2 \\
r &= 2 \\
&\therefore \text{this is a geometric series (with r } > 1) \\
S_1 &= 2 \\
S_2 &= 2 + 4 = 6 \\
S_n &= \frac{a(r^n - 1)}{r - 1} \\
S_{10} &= \frac{2\left((2)^{10} - 1\right)}{2 - 1} \\
&= 2\left((2)^{10} - 1\right) \\
&= 2046 \\
S_{100} &= \frac{2\left((2)^{100} - 1\right)}{2 - 1} \\
&= 2\left((2)^{100} - 1\right) \\
&= 2{,}5 \times 10^{30} \\
S_n &= \frac{2((2)^n - 1)}{2 - 1} \\
&= (2)^{n+1} - 2 \\
\therefore S_n &\to \infty \text{ as } n \to \infty
\end{aligned}
$$

This is a divergent series.

Check answers online with the exercise code below or click on 'show me the answer'.
1. 287J 2. 287K 3. 287M 4. 287N

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Infinite geometric series

Exercise 1 – 11: Sum to infinity

1. What value does $\left(\frac{2}{5}\right)^n$ approach as n tends towards ∞?

Solution:

$$
\begin{aligned}
S_n &= \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \dots \\
\therefore a &= \frac{2}{5} \\
\text{And } r &= \frac{\frac{4}{25}}{\frac{2}{5}} \\
&= \frac{2}{5} \qquad (-1 < r < 1) \\
\text{So then } S_\infty &= \frac{a}{1-r} \\
&= \frac{\frac{2}{5}}{1-\frac{2}{5}} \\
&= \frac{\frac{2}{5}}{\frac{3}{5}} \\
&= \frac{2}{3}
\end{aligned}
$$

2. Find the sum to infinity of the geometric series $3 + 1 + \frac{1}{3} + \frac{1}{9} + \cdots$

   Solution:

$$
\begin{aligned}
a &= 3 \\
r &= \frac{1}{3} \\
S_\infty &= \frac{a}{1-r} \\
&= \frac{3}{1-\frac{1}{3}} \\
&= \frac{3}{\frac{2}{3}} \\
&= \frac{9}{2}
\end{aligned}
$$

3. Determine for which values of x, the geometric series $2 + \frac{2}{3}(x+1) + \frac{2}{9}(x+1)^2 + \cdots$ will converge.

   Solution:

$$
\begin{aligned}
a &= 2 \\
r &= \frac{\frac{2}{3}(x+1)}{2} \\
&= \frac{1}{3}(x+1)
\end{aligned}
$$

   For the series to converge, $-1 < r < 1$, therefore:

$$
\begin{aligned}
-1 < r &< 1 \\
-1 < \frac{1}{3}(x+1) &< 1 \\
-3 < (x+1) &< 3 \\
-3 - 1 < x &< 3 - 1 \\
-4 < x &< 2
\end{aligned}
$$

4. The sum to infinity of a geometric series with positive terms is $4\frac{1}{6}$ and the sum of the first two terms is $2\frac{2}{3}$. Find a, the first term, and r, the constant ratio between consecutive terms.

Solution:

$$\begin{aligned}
T_1 + T_2 &= \frac{8}{3} \\
\therefore a + ar &= \frac{8}{3} \\
a(1+r) &= \frac{8}{3} \\
\therefore a &= \frac{8}{3(1+r)} \ldots\ldots(1) \\
S_\infty &= 4\frac{1}{6} = \frac{25}{6} \\
\therefore \frac{a}{1-r} &= \frac{25}{6} \\
6a &= 25(1-r) \ldots\ldots(2) \\
\text{Substitute eqn. } (1) \to (2): \quad 6\left(\frac{8}{3(1+r)}\right) &= 25(1-r) \\
16 &= 25(1-r)(1+r) \\
16 &= 25(1-r^2) \\
16 &= 25 - 25r^2 \\
25r^2 &= 25 - 16 \\
25r^2 &= 9 \\
r^2 &= \frac{9}{25} \\
\therefore r &= \pm\frac{3}{5} \\
\text{But } T_n &> 0 \\
\therefore r &= \frac{3}{5} \\
\text{And } a &= \frac{8}{3(1+r)} \\
&= \frac{8}{3+3r} \\
&= \frac{8}{3+3\left(\frac{3}{5}\right)} \\
&= \frac{8}{\frac{15}{5}+\frac{9}{5}} \\
&= \frac{8}{\frac{24}{5}} \\
\therefore a &= \frac{5}{3}
\end{aligned}$$

5. Use the sum to infinity to show that $0{,}\dot{9} = 1$.

Solution:

Rewrite the recurring decimal:

$$0{,}\dot{9} = \frac{9}{10} + \frac{9}{100} + + \frac{9}{1000} + \ldots$$

This is a geometric series with $a = \frac{9}{10}$ and $r = \frac{1}{10}$.

$$\begin{aligned}S_\infty &= \frac{a}{1-r}\\ &= \frac{\frac{9}{10}}{1-\frac{1}{10}}\\ &= \frac{\frac{9}{10}}{\frac{9}{10}}\\ &= 1\end{aligned}$$

6. A shrub 110 cm high is planted in a garden. At the end of the first year, the shrub is 120 cm tall. Thereafter the growth of the shrub each year is half of it's growth in the previous year. Show that the height of the shrub will never exceed 130 cm. Draw a graph of the relationship between time and growth.

   [IEB, Nov 2003]

   Solution:

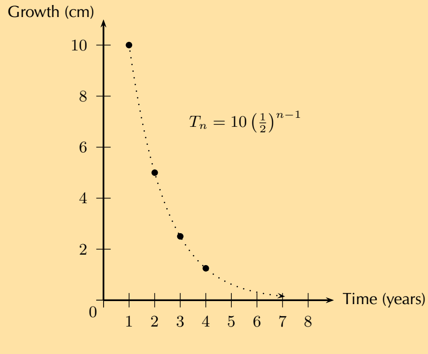
   Write the annual growth of the shrub as a series:

   $$10+5+\frac{5}{2}+\frac{5}{4}+\ldots$$

   This is a geometric series with $a = 10$ and $r = \frac{1}{2}$.

   $$\begin{aligned}S_\infty &= \frac{a}{1-r}\\ &= \frac{10}{1-\frac{1}{2}}\\ &= \frac{10}{\frac{1}{2}}\\ &= 20\end{aligned}$$

   Therefore the growth of the shrub is limited to 20 cm, and the maximum height of the shrub is therefore 110 cm +20 cm = 130 cm.



   Note: we may join the points on the graph because the growth is continuous.

7. Find p:

   $$\sum_{k=1}^{\infty} 27p^k = \sum_{t=1}^{12}(24-3t)$$

   Solution:

   Write out the series on the RHS:

   $$\sum_{t=1}^{12}(24-3t) = 21+18+15+\ldots$$

   This is an arithmetic series with $a = 21$ and $d = -3$.

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ S_{12} &= \frac{12}{2}[2(21) + (12-1)(3)] \\ &= 6[42 - 33] \\ &= 54 \end{aligned}$$

$$\therefore \sum_{t=1}^{12} (24 - 3t) = 54$$

Write out the series on the LHS:

$$\sum_{k=1}^{\infty} 27p^k = 27p + 27p^2 + 27p^3 + \ldots$$

This is a geometric series with $a = 27p$ and $r = p$ $\quad (-1 < p < 1$ for the series to converge).

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ \therefore 54 &= \frac{27p}{1-p} \\ 27p &= 54 - 54p \\ 81p &= 54 \\ \therefore p &= \frac{54}{81} \\ &= \frac{2}{3} \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.
1. 287Q 2. 287R 3. 287S 4. 287T 5. 287V 6. 287W
7. 287X

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1.7 Summary

Exercise 1 – 12: End of chapter exercises

1. Is $1 + 2 + 3 + 4 + \cdots$ an example of a *finite series* or an *infinite series*?

   Solution:

   Infinite arithmetic series

2. A new soccer competition requires each of 8 teams to play every other team once.

   a) Calculate the total number of matches to be played in the competition.

   Solution:

$$7 + 6 + 5 + \ldots$$

$$
\begin{aligned}
a &= 7 \\
d &= -1 \\
S_n &= \frac{n}{2}[2a + (n-1)d] \\
S_7 &= \frac{7}{2}[2(7) + (6)(-1)] \\
&= \frac{7}{2}[8] \\
&= 28
\end{aligned}
$$

$S_7 = 28$

b) If each of n teams played each other once, determine a formula for the total number of matches in terms of n.

Solution:

$$(n-1) + (n-2) + (n-3) + \ldots + 1$$

$$
\begin{aligned}
a &= n - 1 \\
d &= -1 \\
l &= 1 \\
S_n &= \frac{n}{2}[a + l] \\
S_{n-1} &= \frac{n-1}{2}[(n-1) + (1)] \\
&= \frac{n-1}{2}[n] \\
&= \frac{n(n-1)}{2}
\end{aligned}
$$

3. Calculate:

$$\sum_{k=2}^{6} 3\left(\frac{1}{3}\right)^{k+2}$$

Solution:

Number of terms = end index - start index $+1 = (6-2) + 1 = 5$

$$
\begin{aligned}
k = 2: \quad T_1 &= 3\left(\frac{1}{3}\right)^4 = \frac{1}{27} \\
k = 3: \quad T_2 &= 3\left(\frac{1}{3}\right)^5 = \frac{1}{81} \\
\therefore r &= \frac{T_2}{T_1} = \frac{1}{3} \\
a &= \frac{1}{27} \\
\therefore S_n &= \frac{a(1 - r^n)}{1 - r} \\
S_5 &= \frac{\frac{1}{27}\left(1 - \left(\frac{1}{3}\right)^5\right)}{1 - \frac{1}{3}} \\
&= \frac{\frac{1}{27}\left(\frac{242}{243}\right)}{\frac{2}{3}} \\
&= \frac{242}{6561} \times \frac{3}{2} \\
&= \frac{121}{2187}
\end{aligned}
$$

4. The first three terms of a convergent geometric series are: $x + 1; x - 1; 2x - 5$.

a) Calculate the value of x, ($x \neq 1$ or 1).

Solution:

$$
\begin{aligned}
r &= \frac{T_2}{T_1} = \frac{T_3}{T_2} \\
\frac{x-1}{x+1} &= \frac{2x-5}{x-1} \\
\therefore (x-1)^2 &= (2x-5)(x+1) \\
x^2 - 2x + 1 &= 2x^2 - 3x - 5 \\
x^2 - x - 6 &= 0 \\
(x-3)(x+2) &= 0 \\
\therefore x = 3 &\text{ or } x = -2 \\
\text{If } x &= -2 \\
\text{Geometric sequence: } &- 1; -3; -9 \\
\therefore r &= 3 \\
&\therefore \text{will diverge} \\
\text{If } x &= 3 \\
\text{Geometric sequence: } &4; 2; 1 \\
\therefore r &= \frac{1}{2} \\
&\therefore \text{will converge} \\
\therefore x &= 3
\end{aligned}
$$

b) Sum to infinity of the series.

Solution:

$$
\begin{aligned}
4;\ 2;\ 1; \ldots \\
S_\infty &= \frac{a}{1-r} \\
&= \frac{4}{1-\frac{1}{2}} \\
&= \frac{4}{\frac{1}{2}} \\
&= 8
\end{aligned}
$$

5. Write the sum of the first twenty terms of the following series in $\sum$ notation.

$$6 + 3 + \frac{3}{2} + \frac{3}{4} + \cdots$$

Solution:

This is a geometric series:

$$
\begin{aligned}
a &= 6 \\
r &= \frac{1}{2} \\
T_n &= ar^{n-1} \\
&= 6\left(\frac{1}{2}\right)^{n-1} \\
\therefore \sum_{n=1}^{20} 6\left(\frac{1}{2}\right)^{n-1}
\end{aligned}
$$

6. Determine:

$$\sum_{k=1}^{\infty} 12\left(\frac{1}{5}\right)^{k-1}$$

Solution:

$$\sum_{k=1}^{\infty} 12\left(\frac{1}{5}\right)^{k-1}$$

$$\begin{aligned}
&\therefore \text{this is a geometric sequence}\\
r &= \frac{1}{5}\\
&\therefore \text{series converges}\\
a &= 12\\
S_{\infty} &= \frac{a}{1-r}\\
&= \frac{12}{1-\frac{1}{5}}\\
&= \frac{12}{\frac{4}{5}}\\
&= 15
\end{aligned}$$

7. A man was injured in an accident at work. He receives a disability grant of R 4800 in the first year. This grant increases with a fixed amount each year.

   a) What is the annual increase if he received a total of R 143 500 over 20 years?
   Solution:

$$4800 + (4800 + d) + (4800 + 2d) + \ldots$$

   This is an arithmetic series.

$$\begin{aligned}
a &= 4800\\
S_n &= \frac{n}{2}[2a + (n-1)d]\\
S_{20} &= \frac{20}{2}[2(4800) + (20-1)d]\\
\text{And } S_{20} &= 143\ 500\\
\therefore 143\ 500 &= 10[9600 + 19d]\\
14\ 350 &= 9600 + 19d\\
4750 &= 19d\\
\therefore 250 &= d
\end{aligned}$$

   b) His initial annual expenditure is R 2600, which increases at a rate of R 400 per year. After how many years will his expenses exceed his income?
   Solution:

$$2600 + 3000 + 3400 + \ldots$$

   This is an arithmetic series.

$$\begin{aligned}
a &= 2600\\
d &= 400\\
S_n &= \frac{n}{2}[2a + (n-1)d]\\
S_{\text{expenses}} &= \frac{n}{2}[2(2600) + (n-1)(400)]\\
S_e &= \frac{n}{2}[5200 + 400n - 400]\\
&= \frac{n}{2}[4800 + 400n]
\end{aligned}$$

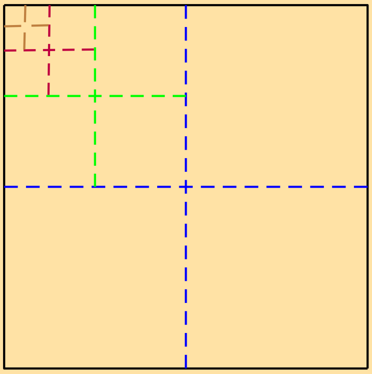
$$S_{\text{income}} = \frac{n}{2}[2(4800) + (n-1)(250)]$$
$$\begin{aligned} S_i &= \frac{n}{2}[2(4800) + (n-1)(250)] \\ &= \frac{n}{2}[9600 + 250n - 250] \\ &= \frac{n}{2}[9350 + 250n] \end{aligned}$$

$$\begin{aligned} \text{let } S_e &= S_i \\ \frac{n}{2}[4800 + 400n] &= \frac{n}{2}[9350 + 250n] \\ 4800 + 400n &= 9350 + 250n \\ 150n &= 4550 \\ \therefore n &= \frac{4550}{150} \\ &= 30{,}333\ldots \end{aligned}$$

Therefore his expenses will exceed his income after 30 years.

8. The length of the side of a square is 4 units. This square is divided into 4 equal, smaller squares. One of the smaller squares is then divided into four equal, even smaller squares. One of the even smaller squares is divided into four, equal squares. This process is repeated indefinitely. Calculate the sum of the areas of all the squares.

Solution:



$$\begin{aligned} \text{After first division of square: Area } &= 2 \times 2 = 4 \\ \text{After second division of square: Area } &= 1 \times 1 = 1 \\ \text{After third division of square: Area } &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \end{aligned}$$

This gives the geometric series:

$$4 + 1 + \frac{1}{4} + \ldots$$

$$\begin{aligned} a &= 4 \\ r &= \frac{1}{4} \\ S_\infty &= \frac{a}{1-r} \\ &= \frac{4}{1-\frac{1}{4}} \\ &= \frac{4}{\frac{3}{4}} \\ &= \frac{16}{3} \end{aligned}$$

9. Thembi worked part-time to buy a Mathematics book which costs R $29{,}50$. On 1 February she saved R $1{,}60$, and every day saves 30 cents more than she saved the previous day. So, on the second day, she saved R $1{,}90$, and so on. After how many days did she have enough money to buy the book?

   Solution:

   $$1{,}60 + 1{,}90 + 2{,}10 + \ldots$$

   This is an arithmetic series.

   $$\begin{aligned} a &= 1{,}60 \\ d &= 0{,}3 \\ S_n &= \frac{n}{2}[2a + (n-1)d] \\ \therefore 29{,}50 &= \frac{n}{2}[2(1{,}60) + (n-1)(0{,}3)] \\ 59 &= n[3{,}2 + 0{,}3n - 0{,}3] \\ 59 &= 2{,}9n + 0{,}3n^2 \\ (\times 10:) \quad 590 &= 29n + 3n^2 \\ 0 &= 3n^2 + 29n - 590 \\ 0 &= (3n+59)(n-10) \\ \therefore n &= -\frac{59}{3} \text{ or } n = 10 \\ \therefore n &= 10 \end{aligned}$$

   since n must be a positive integer.

10. A plant reaches a height of 118 mm after one year under ideal conditions in a greenhouse. During the next year, the height increases by 12 mm. In each successive year, the height increases by $\frac{5}{8}$ of the previous year's growth. Show that the plant will never reach a height of more than 150 mm.

   Solution:

   $$12 + 12\left(\frac{5}{8}\right) + 12\left(\frac{5}{8}\right)^2 + \ldots 12\left(\frac{5}{8}\right)^{n-1}$$

   This is a geometric series.

   $$\begin{aligned} a &= 12 \\ r &= \frac{5}{8} \\ S_\infty &= \frac{a}{1-r} \\ &= \frac{12}{1-\frac{5}{8}} \\ &= \frac{12}{\frac{3}{8}} \\ &= 32 \end{aligned}$$

   Therefore the height of the plant is limited to 118 mm $+$ 32 mm $=$ 150 mm

11. Calculate the value of n if:

   $$\sum_{a=1}^{n}(20 - 4a) = -20$$

Solution:

$$16 + 12 + 8 + \ldots$$

This is an arithmetic series.

$$\begin{aligned} a &= 16 \\ d &= -4 \\ S_n &= \frac{n}{2}[2a + (n-1)d] \\ -20 &= \frac{n}{2}[2(16) + (n-1)(-4)] \\ -40 &= n[32 - 4n + 4] \\ -40 &= 36n - 4n^2 \\ 0 &= -4n^2 + 36n + 40 \\ &= n^2 - 9n - 10 \\ &= (n+1)(n-10) \\ &= (n+1)(n-10) \\ \therefore n &= -1 \text{ or } n = 10 \\ \therefore n &= 10 \end{aligned}$$

since n must be a positive integer.

12. Michael saved R 400 during the first month of his working life. In each subsequent month, he saved 10% more than what he had saved in the previous month.

a) How much did he save in the seventh working month?

Solution:

$$400 + 400(1{,}1) + 400(1{,}1)^2 + \ldots$$

This is a geometric series.

$$\begin{aligned} a &= 400 \\ r &= 1{,}1 \\ T_n &= ar^{n-1} \\ \therefore T_7 &= 400(1{,}1)^6 \\ &= 708{,}62 \end{aligned}$$

Therefore he saved R 708,62 in seventh month.

b) How much did he save all together in his first 12 working months?

Solution:

$$\begin{aligned} S_{12} &= \frac{a(1-r^n)}{1-r} \\ &= \frac{400(1-(1{,}1)^{12})}{1-1{,}1} \\ &= 8553{,}71 \end{aligned}$$

Therefore he saved R 8553,71 in first 12 months.

13. The Cape Town High School wants to build a school hall and is busy with fundraising. Mr. Manuel, an ex-learner of the school and a successful politician, offers to donate money to the school. Having enjoyed mathematics at school, he decides to donate an amount of money on the following basis. He sets a mathematical quiz with 20 questions. For the correct answer to the first question (any learner may answer), the school will receive R 1, for a correct answer to the second question, the school will receive R 2, and so on. The donations $1; 2; 4; \ldots$ form a geometric sequence. Calculate, to the nearest Rand:

a) The amount of money that the school will receive for the correct answer to the 20^{th} question.

Solution:

$$1 + 2 + 4 + \ldots$$

This is a geometric series.

$$\begin{aligned} a &= 1 \\ r &= 2 \\ T_n &= ar^{n-1} \\ \therefore T_{20} &= (1)\left(2^{19}\right) \\ &= 524\ 288 \end{aligned}$$

They receive R $524\ 288$ for question 20.

b) The total amount of money that the school will receive if all 20 questions are answered correctly.

Solution:

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{(1)[2^{20} - 1]}{2 - 1} \\ &= 1\ 048\ 575 \end{aligned}$$

They receive R $1\ 048\ 575$!

14. The first term of a geometric sequence is 9, and the ratio of the sum of the first eight terms to the sum of the first four terms is $97 : 81$. Find the first three terms of the sequence, if it is given that all the terms are positive.

Solution:

$$\begin{aligned} S_n &= \frac{a(1 - r^n)}{1 - r} \\ a &= 9 \\ \frac{S_8}{S_4} &= \frac{97}{81} \\ &= \frac{\frac{9(1-r^8)}{1-r}}{\frac{9(1-r^4)}{1-r}} \\ &= \frac{1 - r^8}{1 - r^4} \\ \therefore \frac{97}{81} &= \frac{1 - r^8}{1 - r^4} \\ \frac{97}{81} &= \frac{(1 - r^4)(1 + r^4)}{1 - r^4} \\ \frac{97}{81} &= 1 + r^4 \\ \frac{16}{81} &= r^4 \\ \therefore r &= \pm\frac{2}{3} \\ \text{But terms are positive } \therefore r &= \frac{2}{3} \end{aligned}$$

$$9; 6; 4; \ldots$$

15. Given the geometric sequence: $6 + p; 10 + p; 15 + p$

a) Determine p, ($p \neq -6$ or -10).

Solution:

$$\begin{aligned}
\frac{10+p}{6+p} &= \frac{15+p}{10+p} \\
\therefore (10+p)(10+p) &= (15+p)(6+p) \\
100 + 20p + p^2 &= 90 + 21p + p^2 \\
\therefore p &= 10
\end{aligned}$$

b) Show that the constant ratio is $\frac{5}{4}$.

Solution:

$$\begin{aligned}
r &= \frac{10+p}{6+p} \\
&= \frac{20}{16} \\
&= \frac{5}{4}
\end{aligned}$$

c) Determine the tenth term of this sequence correct to one decimal place.

Solution:

$$\begin{aligned}
T_n &= ar^{n-1} \\
T_{10} &= (16)\left(\frac{5}{4}\right)^9 \\
&= 119{,}2
\end{aligned}$$

$T_{10} = 119{,}2$

16. The second and fourth terms of a convergent geometric series are 36 and 16, respectively. Find the sum to infinity of this series, if all its terms are positive.

Solution:

$$\begin{aligned}
\frac{T_4}{T_2} &= \frac{ar^3}{ar} \\
&= \frac{16}{36} \\
\therefore r^2 &= \frac{16}{36} \\
r &= \pm\frac{2}{3} \\
\text{But terms are positive, } r &= \frac{2}{3} \\
T_2 &= ar \\
&= 36
\end{aligned}$$

$$\begin{aligned}
a\left(\frac{2}{3}\right) &= 36 \\
\therefore a &= 54 \\
S_\infty &= \frac{a}{1-r} \\
&= \frac{54}{1-\frac{2}{3}} \\
&= 162
\end{aligned}$$

17. Evaluate:

$$\sum_{k=2}^{5} \frac{k\,(k+1)}{2}$$

Solution:

$$\frac{2(2+1)}{2} + \frac{3(3+1)}{2} + \frac{4(4+1)}{2} + \frac{5(5+1)}{2}$$

$$3 + 6 + 10 + 15 = 34$$

18. $S_n = 4n^2 + 1$ represents the sum of the first n terms of a particular series. Find the second term.

Solution:

$$\begin{aligned} S_n &= 4n^2 + 1 \\ S_1 &= 4(1)^2 + 1 = 5 \\ S_2 &= 4(2)^2 + 1 = 17 \\ T_2 &= S_2 - S_1 \\ &= 17 - 5 \\ &= 12 \end{aligned}$$

19. Determine whether the following series converges for the given values of x. If it does converge, calculate the sum to infinity.

$$\sum_{p=1}^{\infty} (x+2)^p$$

a) $x = -\frac{5}{2}$

Solution:

$$\begin{aligned} x &= -\frac{5}{2} \\ \sum_{p=1}^{\infty} (x+2)^p &= \sum_{p=1}^{\infty} \left(-\frac{5}{2} + 2\right)^p \\ &= \sum_{p=1}^{\infty} \left(-\frac{1}{2}\right)^p \end{aligned}$$

$$-\frac{1}{2}; \frac{1}{4}; -\frac{1}{8}; \ldots$$

This is a geometric sequence.

$$r = -\frac{1}{2}$$

$\therefore$ series converges since

$$\begin{aligned} -1 &< r < 1 \\ S_\infty &= \frac{a}{1-r} \\ &= \frac{-\frac{1}{2}}{1 - \left(-\frac{1}{2}\right)} \\ &= \frac{-\frac{1}{2}}{\frac{3}{2}} \\ &= -\frac{1}{3} \end{aligned}$$

b) $x = -5$

Solution:

$$\begin{aligned} x &= -5 \\ \sum_{p=1}^{\infty} (x+2)^p &= \sum_{p=1}^{\infty} (-5+2)^p \\ &= \sum_{p=1}^{\infty} (-3)^p \end{aligned}$$

$$-3; 9; -27; \ldots$$

This is a geometric sequence.

$$\begin{aligned} &r = -3 \\ &\therefore \text{series does not converge since } r < -1 \end{aligned}$$

20. Calculate:

$$\sum_{i=1}^{\infty} 5\left(4^{-i}\right)$$

Solution:

$$\frac{5}{4}; \frac{5}{4^2}; \frac{5}{4^3}; \ldots \frac{5}{4^n}$$

This is a geometric sequence.

$$\begin{aligned} r &= \frac{1}{4} \\ a &= \frac{5}{4} \\ S_{\infty} &= \frac{a}{1-r} \\ &= \frac{\frac{5}{4}}{1-\frac{1}{4}} \\ &= \frac{5}{3} \end{aligned}$$

21. The sum of the first p terms of a sequence is $p\,(p+1)$. Find the tenth term.

Solution:

$$\begin{aligned} p = 10: \quad S_{10} &= 10(10+1) = 110 \\ p = 9: \quad S_9 &= 9(9+1) = 90 \\ \text{And } T_{10} &= S_{10} - S_9 \\ \therefore T_{10} &= 110 - 90 \\ &= 20 \end{aligned}$$

Alternative method:

$$\begin{aligned} p = 1: \quad S_1 &= a = 1(1+1) = 2 \\ p = 2: \quad S_2 &= 2(2+1) = 6 \\ \text{And } S_2 &= T_1 + T_2 = 6 \\ \therefore T_2 &= 6 - 2 = 4 \\ S_3 &= T_1 + T_2 + T_3 = 3 \times 4 = 12 \\ \therefore T_3 &= 12 - 6 = 6 \end{aligned}$$

$$2; 4; 6; \ldots$$

This is an arithmetic sequence.

$$\begin{aligned} a &= 2 \\ d &= 2 \\ T_n &= a + (n-1)d \\ T_{10} &= 2 + (10-1)(2) \\ &= 20 \end{aligned}$$

22. The powers of 2 are removed from the following set of positive integers
$1; 2; 3; 4; 5; 6; \ldots; 1998; 1999; 2000$
Find the sum of remaining integers.

Solution:

$$1; 2^1; 3; 2^2; 5; 6; \ldots 2000$$

$$\begin{aligned} S_n &= \frac{n}{2}[a + l] \\ S_{2000} &= \frac{2000}{2}[1 + 2000] \\ &= 1000(2001) \\ &= 2001000 \end{aligned}$$

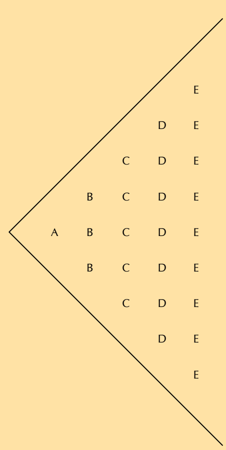
Remove the powers of 2 to make a separate series:

$$2^0; 2^1; 2^2; \ldots 2^{10}$$

This is a geometric series.

$$\begin{aligned} a &= 1 \\ r &= 2 \\ S_n &= \frac{a(r^n - 1)}{r - 1} \\ S_{11} &= \frac{1(2^{11} - 1)}{2 - 1} \\ &= 2047 \\ \therefore S_{2000} - S_{11} &= 2001000 - 2047 \\ &= 1998953 \end{aligned}$$

23. Observe the pattern below:



a) If the pattern continues, find the number of letters in the column containing M's.
Solution:

$$1; 3; 5; 7; \ldots$$

This is an arithmetic series.

$$\begin{aligned} a &= 1 \\ d &= 2 \\ T_n &= a + (n-1)d \\ \text{For the letter M:} \quad n &= 13: \\ T_{13} &= 1 + (12)(2) \\ &= 25 \end{aligned}$$

b) If the total number of letters in the pattern is 361, which letter will be in the last column.
Solution:

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ 361 &= \frac{n}{2}[2a + (n-1)(2)] \\ 361 &= n[1 + n - 1] \\ 361 &= n^2 \\ \therefore n &= \pm 19 \\ \therefore n &= 19 \end{aligned}$$

so the letter "s" will be in the last column.

24. Write $0{,}5\dot{7}$ as a proper fraction.

Solution:
Rewrite the recurring decimal:

$$\begin{aligned} 0{,}5\dot{7} &= 0{,}5 + 0{,}0\dot{7} \\ 0{,}5\dot{7} &= 0{,}5 + [0{,}07 + 0{,}007 + \ldots] \end{aligned}$$

The part in the square brackets is a geometric series with $a = \frac{7}{100}$ and $r = \frac{1}{10}$.

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{\frac{7}{100}}{1 - \frac{1}{10}} \\ &= \frac{\frac{7}{100}}{\frac{9}{10}} \\ &= \frac{7}{90} \\ \therefore 0{,}5\dot{7} &= 0{,}5 + 0{,}0\dot{7} \\ &= \frac{1}{2} + \frac{7}{90} \\ &= \frac{52}{90} \\ &= \frac{26}{45} \end{aligned}$$

25. Given:

$$f(x) = \sum_{p=1}^{\infty} \frac{(1+x)^p}{1-x}$$

a) For which values of x will $f(x)$ converge?

Solution:

$$\frac{(1+x)}{1-x} + \frac{(1+x)^2}{1-x} + \frac{(1+x)^3}{1-x} + \ldots$$

This is a geometric series with $a = \frac{(1+x)}{1-x}$ and $r = (1+x)$. For the series to converge:

$$\begin{aligned} -1 &< r < 1 \\ -1 &< 1+x < 1 \\ -2 &< x < 0 \end{aligned}$$

b) Determine the value of $f\left(-\frac{1}{2}\right)$.

Solution:

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{\frac{1+x}{1-x}}{1-(1+x)} \\ &= \frac{1+x}{(1-x)(-x)} \\ \text{Substitute } x &= -\frac{1}{2} \\ f(-\frac{1}{2}) &= \frac{\frac{1}{2}}{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)} \\ &= \frac{2}{3} \end{aligned}$$

26. From the definition of a geometric sequence, deduce a formula for calculating the sum of n terms of the series

$$a^2 + a^4 + a^6 + \cdots$$

Solution:

$$\begin{aligned} r &= a^2 \\ T_n &= a^{2n} \\ \therefore S_n &= a^2 + a^4 + a^6 + \ldots + a^{2n-2} + a^{2n} \ldots\ldots (1) \\ r \times S_n = a^2 S_n &= \qquad a^4 + a^6 + \ldots + a^{2n} + a^{2n+2} \ldots\ldots (2) \\ (1)-(2): \quad S_n - a^2 S_n &= a^2 - a^{2n+2} \\ S_n(1-a^2) &= a^2 - a^2 \ . \ a^{2n} \\ \therefore S_n &= \frac{a^2(1-a^{2n})}{(1-a^2)} \end{aligned}$$

27. Calculate the tenth term of the series if $S_n = 2n + 3n^2$.

Solution:

$$\begin{aligned} p = 10: \quad S_{10} &= 2(10) + 3(10)^2 \\ &= 20 + 300 = 320 \\ p = 9: \quad S_9 &= 2(9) + 3(9)^2 \\ &= 18 + 243 = 261 \\ \text{And } T_{10} &= S_{10} - S_9 \\ \therefore T_{10} &= 320 - 261 \\ &= 59 \end{aligned}$$

Alternative method:

$$
\begin{aligned}
S_1 &= T_1 = 5 \\
S_2 &= T_1 + T_2 = 16 \\
\therefore T_2 &= 16 - 5 \\
&= 11 \\
S_3 &= 2(3) + 3(3)^2 = 33 \\
\therefore 33 &= T_3 + S_2 \\
\therefore T_3 &= 17 \\
S_n &= 5 + 11 + 17 + \dots \\
d &= 11 - 5 = 6 \\
d &= 17 - 11 = 6 \\
&\therefore \text{this is an arithmetic series} \\
T_{10} &= a + (n-1)d \\
&= 5 + (9)(6) \\
&= 59
\end{aligned}
$$

28. A theatre is filling up at a rate of 4 people in the first minute, 6 people in the second minute, and 8 people in the third minute and so on. After 6 minutes the theatre is half full. After how many minutes will the theatre be full?

[IEB, Nov 2001]

Solution:

$$4 + 6 + 8 + \dots$$

$$
\begin{aligned}
d &= T_2 - T_1 \\
&= 6 - 4 \\
&= 2 \\
d &= T_3 - T_2 \\
&= 8 - 6 \\
&= 2 \\
&\therefore \text{this is an arithmetic series} \\
S_n &= \frac{n}{2}[2a + (n-1)d] \\
S_6 &= \frac{6}{2}[2(4) + (5)(2)] \\
&= 3(18) \\
&= 54 \quad \text{(theatre half full)} \\
\therefore 2 \times 54 &= 108 \quad \text{(theatre full)} \\
S_n &= \frac{n}{2}[2a + (n-1)d] \\
108 &= \frac{n}{2}[2(4) + (n-1)(2)] \\
216 &= n[8 + 2n - 2] \\
0 &= 2n^2 + 6n - 216 \\
0 &= n^2 + 3n - 108 \\
0 &= (n+12)(n-9) \\
\therefore n &= -12 \text{ or } n = 9 \\
&n \text{ must be a positive integer} \\
\therefore n &= 9
\end{aligned}
$$

It takes 9 minutes for the theatre to be full.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 287Y	2a. 287Z	2b. 2882	3. 2883	4a. 2884	4b. 2885
5. 2886	6. 2887	7a. 2888	7b. 2889	8. 288B	9. 288C
10. 288D	11. 288F	12a. 288G	12b. 288H	13a. 288J	13b. 288K
14. 288M	15a. 288N	15b. 288P	15c. 288Q	16. 288R	17. 288S
18. 288T	19a. 288V	19b. 288W	20. 288X	21. 288Y	22. 288Z
23a. 2892	23b. 2893	24. 2894	25. 2895	26. 2896	27. 2897
28. 2898					

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CHAPTER 2

Functions

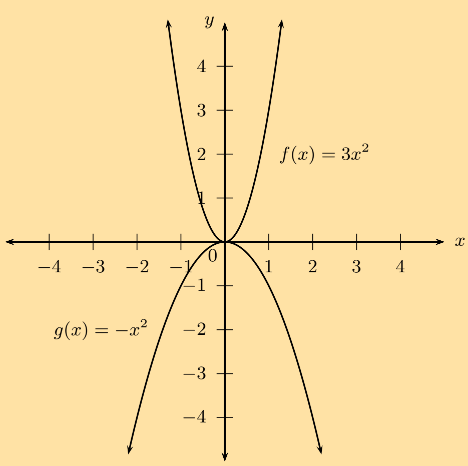
- Learners must be encouraged to check whether or not the inverse is a function.
- It is very important that learners understand that $f^{-1}(x)$ notation can only be used if the inverse is a function.
- Learners must not confuse the inverse function f^{-1} and the reciprocal $\frac{1}{f(x)}$.
- Encourage learners to state restrictions, particularly for quadratic functions.
- Learners must understand that $y = \sqrt{-x}$ has real solutions for $x < 0$.
- Exercises on parabolic functions with horizontal and vertical shifts have been included for enrichment only and are clearly marked.
- The logarithmic function is introduced as the inverse of the exponential function. Learners need to understand that the logarithmic function allows us to rewrite an exponential expression with the exponent as the subject of the formula.
- It is very important that learners can change from exponential form to logarithmic form and vice versa. This skill is also important for finding the period of an investment or loan in the Finance chapter.
- Learners should be encouraged to use the definition and change of base to solve problems. Manipulation involving the logarithmic laws is not examinable.
- Learners should be encouraged to be familiar with the LOG function on their calculator and also to use their calculator to check answers.
- Enrichment content is not examinable and is clearly marked.

2.1 Revision

Exercise 2 – 1: Revision

1. Sketch the graphs on the same set of axes and determine the following for each function:
   - Intercepts
   - Turning point
   - Axes of symmetry
   - Domain and range
   - Maximum and minimum values

   a) $f(x) = 3x^2$ and $g(x) = -x^2$
   Solution:



For $f(x)$:

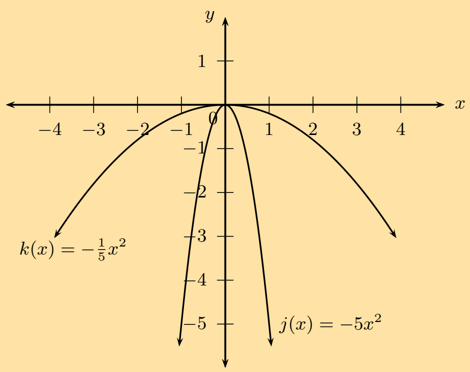
$$\begin{aligned}\text{Intercept: } & (0;0)\\ \text{Turning point: } & (0;0)\\ \text{Axes of symmetry: } & x=0\\ \text{Domain: } & \{x : x \in \mathbb{R}\}\\ \text{Range: } & \{y : y \geq 0, y \in \mathbb{R}\}\\ \text{Minimum value: } & y=0\end{aligned}$$

For $g(x)$:

$$\begin{aligned}\text{Intercept: } & (0;0)\\ \text{Turning point: } & (0;0)\\ \text{Axes of symmetry: } & x=0\\ \text{Domain: } & \{x : x \in \mathbb{R}\}\\ \text{Range: } & \{y : y \leq 0, y \in \mathbb{R}\}\\ \text{Maximum value: } & y=0\end{aligned}$$

b) $j(x) = -\frac{1}{5}x^2$ and $k(x) = -5x^2$

Solution:



For $j(x)$:

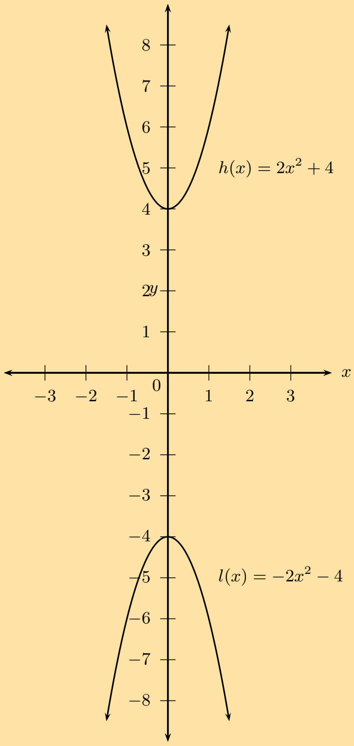
$$\begin{aligned}\text{Intercepts: } & (0;0)\ \ (0;0)\\ \text{Turning point: } & (0;0)\\ \text{Axes of symmetry: } & x=0\\ \text{Domain: } & \{x : x \in \mathbb{R}\}\\ \text{Range: } & \{y : y \leq 0, y \in \mathbb{R}\}\\ \text{Maximum value: } & y=0\end{aligned}$$

For $k(x)$:

$$\begin{aligned}\text{Intercepts: } & (0;0)\ \ (0;0)\\ \text{Turning point: } & (0;0)\\ \text{Axes of symmetry: } & x=0\\ \text{Domain: } & \{x : x \in \mathbb{R}\}\\ \text{Range: } & \{y : y \leq 0, y \in \mathbb{R}\}\\ \text{Maximum value: } & y=0\end{aligned}$$

c) $h(x) = 2x^2 + 4$ and $l(x) = -2x^2 - 4$

Solution:



For $h(x)$:

$$\begin{aligned}\text{Intercept: } & (0;4)\\ \text{Turning point: } & (0;4)\\ \text{Axes of symmetry: } & x = 0\\ \text{Domain: } & \{x : x \in \mathbb{R}\}\\ \text{Range: } & \{y : y \geq 4, y \in \mathbb{R}\}\\ \text{Minimum value: } & y = 4\end{aligned}$$

For $l(x)$:

$$\begin{aligned}\text{Intercept: } & (0;-4)\\ \text{Turning point: } & (0;-4)\\ \text{Axes of symmetry: } & x = 0\\ \text{Domain: } & \{x : x \in \mathbb{R}\}\\ \text{Range: } & \{y : y \leq -4, y \in \mathbb{R}\}\\ \text{Maximum value: } & y = -4\end{aligned}$$

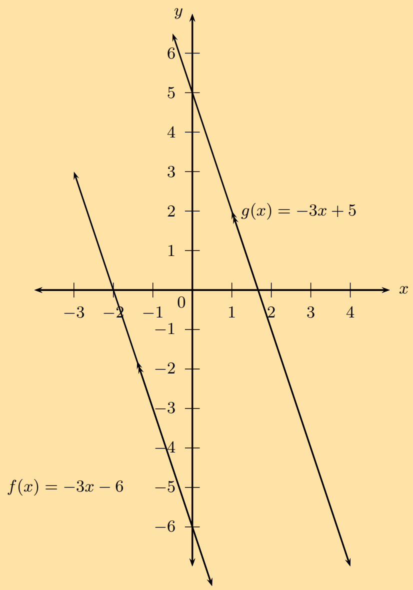
2. Given $f(x) = -3x - 6$ and $g(x) = mx + c$. Determine the values of m and c if $g \parallel f$ and g passes through the point $(1; 2)$. Sketch both functions on the same system of axes.

   Solution:

$$\begin{aligned}g(x) &= mx + c\\ m &= -3\\ g(x) &= -3x + c\\ \text{Substitute } (1;2) \quad 2 &= -3(1) + c\\ \therefore c &= 5\\ \therefore g(x) &= -3x + 5\end{aligned}$$

$$\text{Intercepts for } g: \quad (\frac{5}{3}; 0); (0; 5)$$

$$\text{Intercepts for } f: \quad (-2; 0); (0; -6)$$



3. Given $m : \frac{x}{2} - \frac{y}{3} = 1$ and $n : -\frac{y}{3} = 1$. Determine the x- and y-intercepts and sketch both graphs on the same system of axes.

Solution:

For $m(x)$:

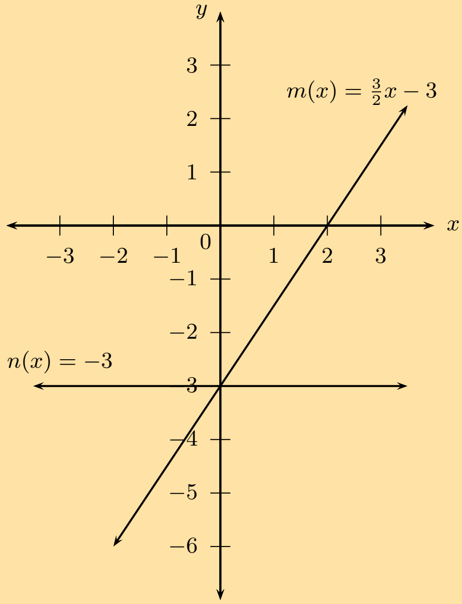
$$\begin{aligned} \frac{x}{2} - \frac{y}{3} &= 1 \\ \text{Let } x = 0: \quad -\frac{y}{3} &= 1 \\ y &= -3 \\ \text{Let } y = 0: \quad \frac{x}{2} &= 1 \\ x &= 2 \\ \text{Intercepts: } &(2; 0); (0; -3) \end{aligned}$$

For $n(x)$:

$$\begin{aligned} -\frac{y}{3} &= 1 \\ \therefore y &= -3 \\ \text{Intercepts: } &(0; -3) \end{aligned}$$

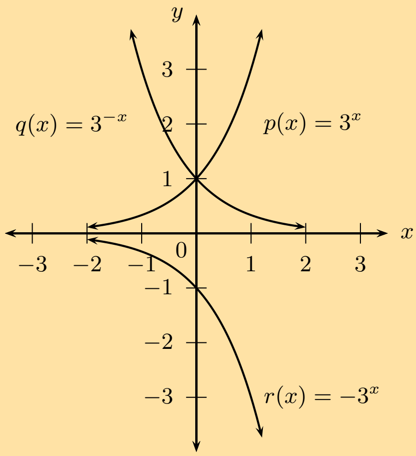
Write the linear function in standard form:

$$\begin{aligned} \frac{x}{2} - \frac{y}{3} &= 1 \\ \frac{3x}{2} - y &= 3 \\ \frac{3x}{2} - 3 &= y \\ \therefore y &= \frac{3x}{2} - 3 \end{aligned}$$



4. Given $p(x) = 3^x, q(x) = 3^{-x}$ and $r(x) = -3^x$.

   a) Sketch p, q and r on the same system of axes.
   Solution:



   b) For each of the functions, determine the intercepts, asymptotes, domain and range.
   Solution:
   For $p(x)$:

Intercept: $(0; 1)$
Asymptote: $y = 0$
Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y > 0\}$

For $q(x)$:

Intercept: $(0; 1)$
Asymptote: $y = 0$
Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y > 0\}$

For $r(x)$:

Intercept: $(0; -1)$
Asymptote: $y = 0$
Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y < 0\}$

Check answers online with the exercise code below or click on 'show me the answer'.
1a. 2899 1b. 289B 1c. 289C 2. 289D 3. 289F 4a. 289G
4b. 289H

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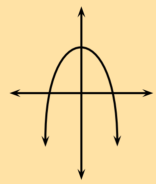
2.2 Functions and relations

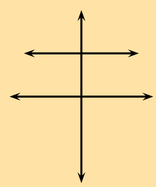
Exercise 2 – 2: Identifying functions

1. Consider the graphs given below and determine whether or not they are functions:



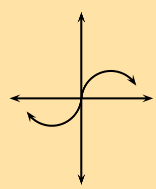
a)

Solution:
Yes



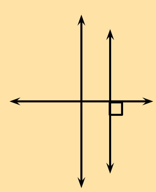
b)

Solution:
Yes



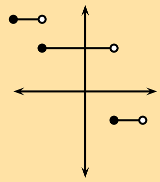
c)

Solution:
Yes



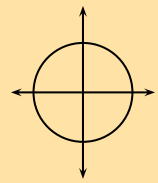
d)

Solution:
No



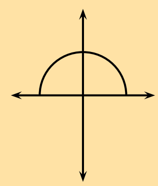
e)

Solution:
Yes



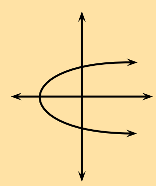
f)

Solution:
No



g)

Solution:
Yes

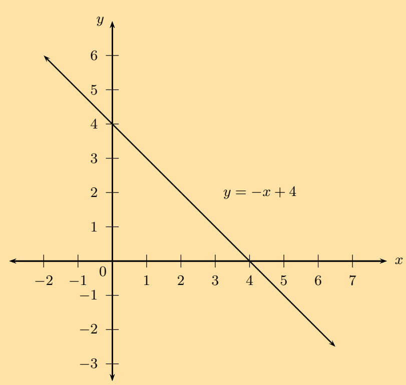


h)

Solution:
No

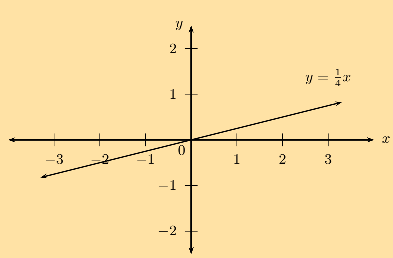
2. Sketch the following and determine whether or not they are functions:

   a) $x + y = 4$
   Solution:



   One-to-one relation: therefore is a function.

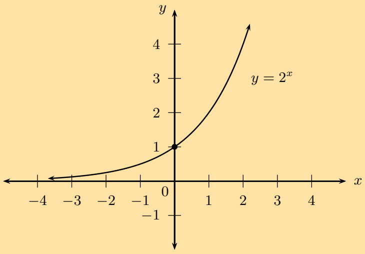
   b) $y = \frac{x}{4}$
   Solution:



One-to-one relation: therefore is a function.

c) $y = 2^x$

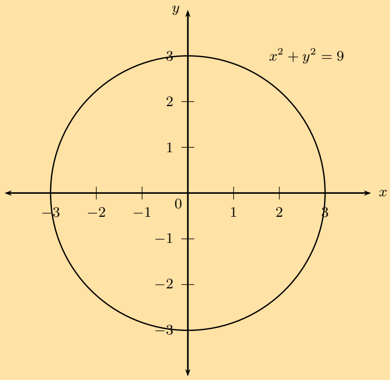
Solution:



One-to-one relation: therefore is a function.

d) $x^2 + y^2 = 9$

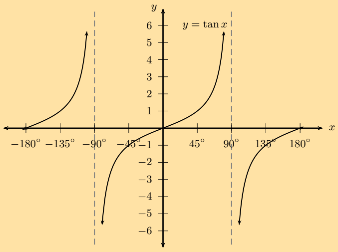
Solution:



One-to-many relation: therefore is not a function.

e) $y = \tan x$

Solution:



Many-to-one relation: therefore is a function.

3. The table below gives the average per capita income, d, in a region of the country as a function of u, the percentage of unemployed people. Write down an equation to show that the average income is a function of the percentage of unemployed people.

u	1	2	3	4
d	22 500	22 000	21 500	21 000

Solution:

Per capita income is a measure of the average amount of money earned per person in a certain area.

We see that there is a constant difference of -500 between the consecutive values of d, therefore the relation is a linear function of the form $y = mx + c$:

u is the independent variable and d is the dependent variable.

$$d = mu + c$$
$$m = -500$$
$$d = -500u + c$$

Substitute any of the given set of values to solve for c:

$$22\ 500 = -500(1) + c$$
$$\therefore c = 23\ 000$$

The function is: $d = -500u + 23\ 000$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 289J 1b. 289K 1c. 289M 1d. 289N 1e. 289P 1f. 289Q
1g. 289R 1h. 289S 2a. 289T 2b. 289V 2c. 289W 2d. 289X
2e. 289Y 3. 289Z

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2.3 Inverse functions

2.4 Linear functions

Inverse of the function $y = ax + q$

Exercise 2 – 3: Inverse of the function $y = ax + q$

1. Given $f(x) = 5x + 4$, find $f^{-1}(x)$.

   Solution:

$$\begin{aligned} f(x) : y &= 5x + 4 \\ f^{-1}(x) : x &= 5y + 4 \\ \therefore x - 4 &= 5y \\ y &= \frac{1}{5}x - \frac{4}{5} \\ \therefore f^{-1}(x) &= \frac{1}{5}x - \frac{4}{5} \end{aligned}$$

2. Consider the relation $f(x) = -3x - 7$.

   a) Is the relation a function? Explain your answer.

   Solution:

   It is a function. Every x-value relates to only one y-value, it is a one-to-one relation.

   b) Identify the domain and range.

   Solution:

$$\text{Domain } \{x : x \in \mathbb{R}\}$$
$$\text{Range } \{y : y \in \mathbb{R}\}$$

   c) Determine $f^{-1}(x)$.

Solution:

$$
\begin{aligned}
f(x): y &= -3x - 7 \\
f^{-1}(x): x &= -3y - 7 \\
\therefore x + 7 &= -3y \\
y &= -\frac{1}{3}x - \frac{7}{3} \\
\therefore f^{-1}(x) &= -\frac{1}{3}x - \frac{7}{3}
\end{aligned}
$$

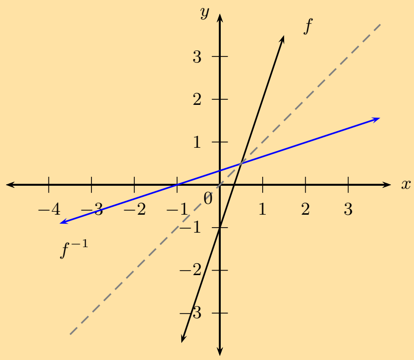
3. a) Sketch the graph of the function $f(x) = 3x - 1$ and its inverse on the same system of axes. Indicate the intercepts and the axis of symmetry of the two graphs.

Solution:

$$
\begin{aligned}
y &= 3x - 1 \\
x &= 3y - 1 \\
\therefore 3y &= x + 1 \\
y &= \frac{1}{3}x + \frac{1}{3}
\end{aligned}
$$

The intercepts are:

$$
\begin{aligned}
f(x): x = 0, y &= -1 \\
y = 0, x &= \frac{1}{3} \\
f^{-1}(x): x = 0, y &= \frac{1}{3} \\
y = 0, x &= -1
\end{aligned}
$$



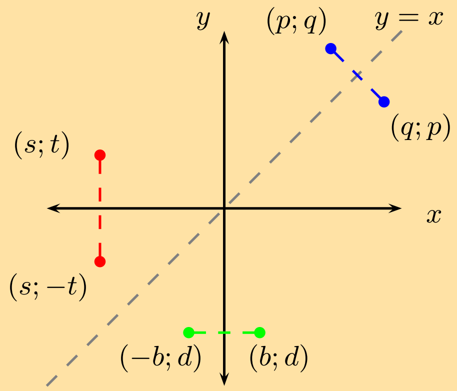
b) $T\left(\frac{4}{3}; 3\right)$ is a point on f and R is a point on f^{-1}. Determine the coordinates of R if R and T are symmetrical.

Solution: $R\left(3; \frac{4}{3}\right)$

4. a) Explain why the line $y = x$ is an axis of symmetry for a function and its inverse.

Solution:

To reflect a function about the y-axis, we replace every x with $-x$. Similarly, to reflect a function about the x-axis, we replace every y with $-y$. To reflect a function about the line $y = x$, we replace x with y and y with x, which is how we determine the inverse.



b) Will the line $y = -x$ be an axis of symmetry for a function and its inverse?
Solution: No it will not.

5. a) Given $f^{-1}(x) = -2x + 4$, determine $f(x)$.
Solution:

$$\begin{aligned} f^{-1}(x) : y &= -2x + 4 \\ f(x) : x &= -2y + 4 \\ \therefore 2y &= -x + 4 \\ y &= -\frac{1}{2}x + 2 \\ \therefore f(x) &= -\frac{1}{2}x + 2 \end{aligned}$$

b) Calculate the intercepts of $f(x)$ and $f^{-1}(x)$.
Solution:
Consider $f(x)$:

$$\begin{aligned} \text{Let } y &= -\frac{1}{2}x + 2 \\ \text{Let } x = 0 : \quad y &= -\frac{1}{2}(0) + 2 \\ y &= 2 \\ \therefore y - &\text{intercept is } (0;2) \end{aligned}$$

$$\begin{aligned} \text{Let } y = 0 : \quad 0 &= -\frac{1}{2}x + 2 \\ -2 &= -\frac{1}{2}x \\ x &= 4 \\ \therefore x - &\text{intercept is } (4;0) \end{aligned}$$

Consider $f^{-1}(x)$:

$$\begin{aligned} \text{Let } y &= -2x + 4 \\ \text{Let } x = 0 : \quad y &= -2(0) + 4 \\ y &= 4 \\ \therefore y - &\text{intercept is } (0;4) \end{aligned}$$

$$\begin{aligned} \text{Let } y = 0 : \quad 0 &= -2x + 4 \\ 2x &= 4 \\ x &= 2 \\ \therefore x - &\text{intercept is } (2;0) \end{aligned}$$

Therefore the intercepts for $f(x)$ are $(4;0)$ and $(0;2)$ and the intercepts for $f^{-1}(x)$ are $(2;0)$ and $(0;4)$.

c) Determine the coordinates of T, the point of intersection of $f(x)$ and $f^{-1}(x)$.

Solution:

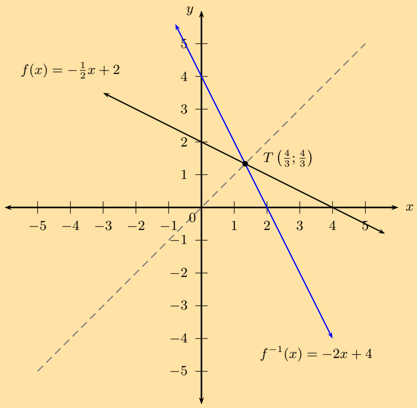
To find the point on intersection, we let $f(x) = f^{-1}(x)$:

$$\begin{aligned} -\frac{1}{2}x + 2 &= -2x + 4 \\ -\frac{1}{2}x + 2x &= 4 - 2 \\ \frac{3}{2}x &= 2 \\ \therefore x &= \frac{4}{3} \\ \text{And } y &= -2\left(\frac{4}{3}\right) + 4 \\ &= -\frac{8}{3} + 4 \\ &= \frac{4}{3} \end{aligned}$$

This gives the point $T\left(\frac{4}{3}; \frac{4}{3}\right)$.

d) Sketch the graphs of f and f^{-1} on the same system of axes. Indicate the intercepts and point T on the graph.

Solution:



e) Is f^{-1} an increasing or decreasing function?

Solution:

Decreasing function. The function values decrease as x increases.

Check answers online with the exercise code below or click on 'show me the answer'.

1\. 28B4 2a. 28B5 2b. 28B6 2c. 28B7 3. 28B8 4. 28B9
5\. 28BB

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Inverse of the function $y = ax^2$

Exercise 2 – 4: Inverses - domain, range, intercepts, restrictions

1. Determine the inverse for each of the following functions:

   a) $y = \frac{3}{4}x^2$
   Solution:

$$\begin{aligned} y &= \frac{3}{4}x^2 \\ \text{Interchange } x \text{ and } y: \quad x &= \frac{3}{4}y^2 \\ \frac{4}{3}x &= y^2 \\ \therefore y &= \pm\sqrt{\frac{4}{3}x} \qquad (x \geq 0) \end{aligned}$$

   b) $4y - 8x^2 = 0$
   Solution:

$$\begin{aligned} \text{Interchange } x \text{ and } y: \quad 4x - 8y^2 &= 0 \\ 4x &= 8y^2 \\ \frac{1}{2}x &= y^2 \\ \therefore y &= \pm\sqrt{\frac{1}{2}x} \qquad (x \geq 0) \end{aligned}$$

   c) $x^2 + 5y = 0$
   Solution:

$$\begin{aligned} \text{Interchange } x \text{ and } y: \quad y^2 + 5x &= 0 \\ y^2 &= -5x \\ \therefore y &= \pm\sqrt{-5x} \qquad (x \leq 0) \end{aligned}$$

   d) $4y - 9 = (x + 3)(x - 3)$
   Solution:

$$\begin{aligned} 4y - 9 &= x^2 - 9 \\ 4y &= x^2 \\ \text{Interchange } x \text{ and } y: \quad 4x &= y^2 \\ \therefore y &= \pm\sqrt{4x} \qquad (x \geq 0) \end{aligned}$$

2. Given the function $g(x) = \frac{1}{2}x^2$ for $x \geq 0$.

   a) Find the inverse of g.

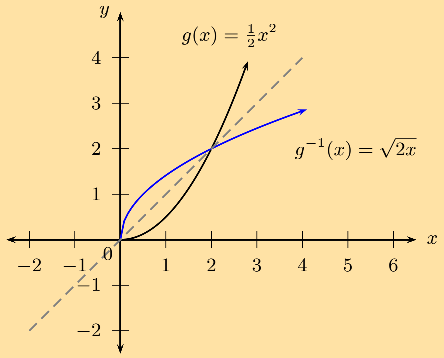
Solution:

$$\begin{aligned}
\text{Let } y &= \frac{1}{2}x^2 \qquad (x \geq 0) \\
\text{Interchange } x \text{ and } y: \quad x &= \frac{1}{2}y^2 \qquad (y \geq 0) \\
2x &= y^2 \\
y &= \sqrt{2x} \qquad (x \geq 0, y \geq 0)
\end{aligned}$$

$$\therefore g^{-1}(x) = \sqrt{2x} \qquad (x \geq 0)$$

b) Draw g and g^{-1} on the same set of axes.

Solution:



c) Is g^{-1} a function? Explain your answer.

Solution:

Yes. It passes the vertical line test and is a one-to-one relation.

d) State the domain and range for g and g^{-1}.

Solution:

$$\begin{aligned}
g: &\quad \text{domain } x \geq 0 \quad \text{range } y \geq 0 \\
g^{-1}: &\quad \text{domain } x \geq 0 \quad \text{range } y \geq 0
\end{aligned}$$

e) Determine the coordinates of the point(s) of intersection of the function and its inverse.

Solution:

To find the points of intersection, we equate $g(x)$ and $g^{-1}(x)$:

$$\begin{aligned}
\frac{1}{2}x^2 &= \sqrt{2x} \\
\left(\frac{1}{2}x^2\right)^2 &= \left(\sqrt{2x}\right)^2 \\
\frac{1}{4}x^4 &= 2x \\
x^4 &= 8x \\
x^4 - 8x &= 0 \\
x(x^3 - 8) &= 0 \\
x(x-2)(x^2+2x+4) &= 0 \\
\therefore x = 0 \text{ or } x = 2 &\text{ or } x^2 + 2x + 4 = 0
\end{aligned}$$

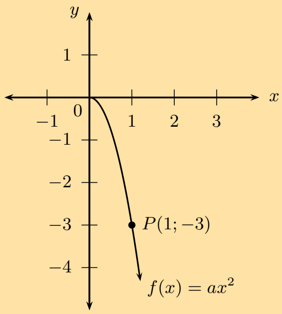
$$\begin{aligned}
\text{If } x &= 0 \\
y &= 0 \\
\text{If } x &= 2 \\
y &= \frac{1}{2}(2)^2 \\
&= 2
\end{aligned}$$

$$\text{If } x^2 + 2x + 4 = 0$$

$$\begin{aligned}
\text{Use the quadratic formula } x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)} \\
&= \frac{-2 \pm \sqrt{-12}}{2} \\
&= \text{no real solution}
\end{aligned}$$

Therefore, we get the points $(0;0)$ or $(2;2)$.

3. Given the graph of the parabola $f(x) = ax^2$ with $x \geq 0$ and passing through the point $P(1;-3)$.



a) Determine the equation of the parabola.

Solution:

$$\begin{aligned}
\text{Let } y &= ax^2 \\
\text{Substitute } P(1;-3): \quad -3 &= a(1)^2 \\
\therefore a &= -3 \\
\therefore f(x) &= -3x^2 \quad (x \geq 0)
\end{aligned}$$

b) State the domain and range of f.

Solution:

$f:$ domain $\{x : x \geq 0\}$ range $\{y : y \leq 0\}$

c) Give the coordinates of the point on f^{-1} that is symmetrical to the point P about the line $y = x$.

Solution: $(-3;1)$

d) Determine the equation of f^{-1}.

Solution:

$$\begin{aligned}
\text{Let } y &= -3x^2 \qquad (x \geq 0) \\
\text{Interchange } x \text{ and } y: \quad x &= -3y^2 \qquad (y \geq 0) \\
-\frac{1}{3}x &= y^2 \\
y &= \sqrt{-\frac{1}{3}x} \qquad (x \leq 0, y \geq 0) \\
\therefore f^{-1}(x) &= \sqrt{-\frac{1}{3}x} \qquad (x \leq 0)
\end{aligned}$$

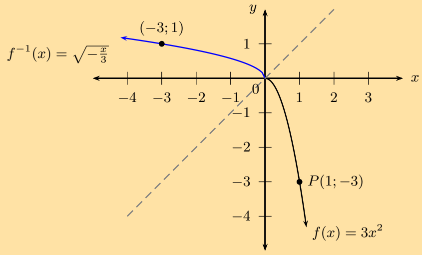
e) State the domain and range of f^{-1}.

Solution:

$f^{-1}:$ domain $\{x : x \leq 0\}$ range $\{y : y \geq 0\}$

f) Draw a graph of f^{-1}.

Solution:



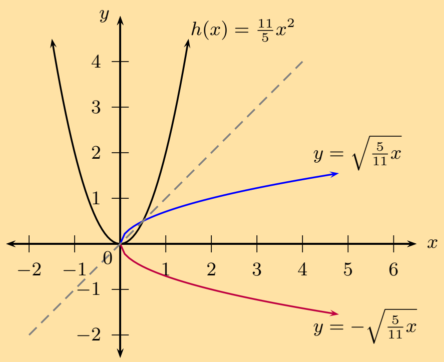
4. a) Determine the inverse of $h(x) = \frac{11}{5}x^2$.

Solution:

$$\text{Let } y = \frac{11}{5}x^2$$
$$\text{Interchange } x \text{ and } y: \quad x = \frac{11}{5}y^2$$
$$\frac{5}{11} = y^2$$
$$\therefore y = \pm\sqrt{\frac{5}{11}x} \qquad (x \geq 0)$$

b) Sketch both graphs on the same system of axes.

Solution:

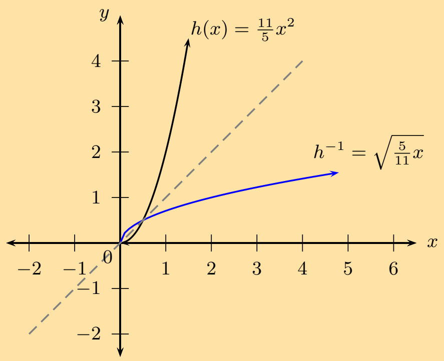


c) Restrict the domain of h so that the inverse is a function.

Solution:

Option 1: Restrict the domain of h to $x \geq 0$ so that the inverse will also be a function (h^{-1}). The restriction $x \geq 0$ on the domain of h will restrict the range of h^{-1} such that $y \geq 0$.

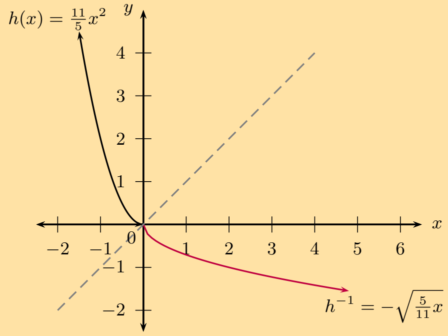
h :	domain $x \geq 0$	range $y \geq 0$
h^{-1} :	domain $x \geq 0$	range $y \geq 0$



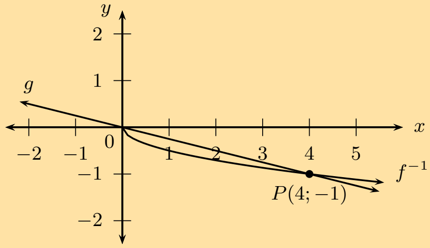
or

Option 2: Restrict the domain of h to $x \leq 0$ so that the inverse will also be a function $\left(h^{-1}\right)$. The restriction $x \leq 0$ on the domain of h will restrict the range of h^{-1} such that $y \leq 0$.

$$\begin{array}{lll} h: & \text{domain } x \leq 0 & \text{range } y \geq 0 \\ h^{-1}: & \text{domain } x \geq 0 & \text{range } y \leq 0 \end{array}$$



5. The diagram shows the graph of $g(x) = mx + c$ and $f^{-1}(x) = a\sqrt{x}$, $(x \geq 0)$. Both graphs pass through the point $P(4; -1)$.



a) Determine the values of a, c and m.

Solution:

From the graph we see that the straight line passes through the origin, therefore $c = 0$.

$$\begin{aligned} g(x) &= mx \\ \text{Substitute } P(4; -1): \quad -1 &= 4m \\ \therefore m &= -\frac{1}{4} \\ \therefore g(x) &= -\frac{1}{4}x \end{aligned}$$

$$\begin{aligned} f^{-1}(x) &= a\sqrt{x} \\ \text{Substitute } P(4; -1): \quad -1 &= a\sqrt{4} \\ -1 &= 2a \\ \therefore a &= -\frac{1}{2} \\ \therefore f^{-1}(x) &= -\frac{1}{2}\sqrt{x} \end{aligned}$$

b) Give the domain and range of f^{-1} and g.

Solution:

$$\begin{array}{lll} g: & \text{domain: } x \in \mathbb{R} & \text{range: } y \in \mathbb{R} \\ f^{-1}: & \text{domain: } x \geq 0 & \text{range: } y \leq 0 \end{array}$$

c) For which values of x is $g(x) < f(x)$?

Solution:

$x > 4$

d) Determine f.

Solution:

$$\begin{aligned} \text{Let } y &= -\frac{1}{2}\sqrt{x} \\ \text{Interchange } x \text{ and } y: \quad x &= -\frac{1}{2}\sqrt{y} \qquad (y \geq 0) \\ -2x &= \sqrt{y} \\ \therefore y &= 4x^2 \qquad (y \geq 0) \end{aligned}$$

e) Determine the coordinates of the point(s) of intersection of g and f intersect.

Solution:

To determine the coordinates of the point(s) of intersection, we equate g and f:

$$\begin{aligned} -\frac{1}{4}x &= 4x^2 \\ 0 &= 4x^2 + \frac{1}{4}x \\ 0 &= x\left(4x + \frac{1}{4}\right) \\ \therefore x &= 0 \text{ or } 4x + \frac{1}{4} = 0 \\ \text{If } x = 0: \quad y &= 0 \\ \text{If } 4x + \frac{1}{4} &= 0: \\ 4x &= -\frac{1}{4} \\ x &= -\frac{1}{16} \\ \text{If } x = -\frac{1}{16}: \quad y &= -\frac{1}{4}\left(-\frac{1}{16}\right) \\ &= \frac{1}{64} \end{aligned}$$

Therefore, the two graphs intersect at $(0; 0)$ and $\left(-\frac{1}{16}; \frac{1}{64}\right)$.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28BC 1b. 28BD 1c. 28BF 1d. 28BG 2. 28BH 3. 28BJ
4. 28BK 5. 28BM

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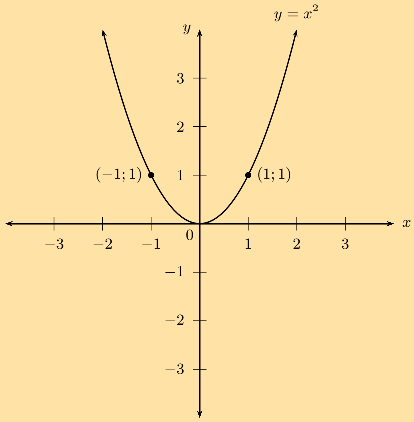
Exercise 2 – 5: Inverses - average gradient, increasing and decreasing functions

1. a) Sketch the graph of $y = x^2$ and label a point other than the origin on the graph.

Solution:



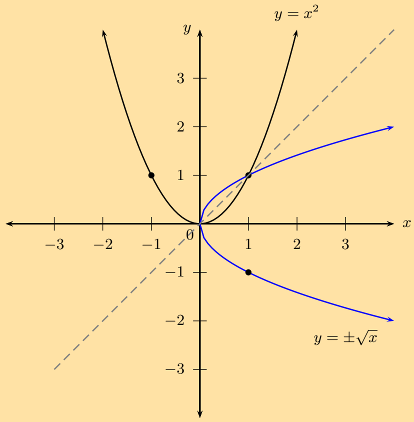
b) Find the equation of the inverse of $y = x^2$.

Solution:

$$y = x^2$$
$$\text{Inverse: } x = y^2$$
$$\therefore y = \pm\sqrt{x} \qquad (x \geq 0)$$

c) Sketch the graph of the inverse on the same system of axes.

Solution:



d) Is the inverse a function? Explain your answer.

Solution:

No. For certain values of x, the inverse cuts a vertical line in two places. Therefore, it is not function.

e) $P(2;4)$ is a point on $y = x^2$. Determine the coordinates of Q, the point on the graph of the inverse which is symmetrical to P about the line $y = x$.

Solution:

$Q(4;2)$

f) Determine the average gradient between:

i. the origin and P;
ii. the origin and Q.

Interpret the answers.

Solution:

i.

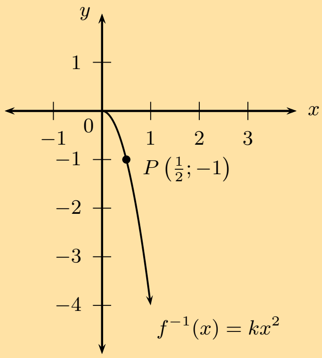
$$\text{Average gradient: } = \frac{y_P - y_O}{x_P - x_O} = \frac{4-0}{2-0} = 2$$

ii.

$$\begin{aligned}\text{Average gradient: } &= \frac{y_Q - y_O}{x_Q - x_O} \\ &= \frac{2-0}{4-0} \\ &= \frac{1}{2}\end{aligned}$$

Average gradient$_{OP} = 2$ and average gradient$_{OQ} = \frac{1}{2}$.
Both gradients are positive, and they are also reciprocals of each other.

2. Given the function $f^{-1}(x) = kx^2$, $x \geq 0$, which passes through the point $P\left(\frac{1}{2}; -1\right)$.



a) Find the value of k.
Solution:

$$\begin{aligned} f^{-1}(x) &= kx^2 \\ \text{Substitute } \left(\frac{1}{2}; -1\right) \quad -1 &= k\left(\frac{1}{2}\right)^2 \\ -1 &= k\left(\frac{1}{4}\right) \\ -4 &= k \\ \therefore f^{-1}(x) &= -4x^2 \end{aligned}$$

b) State the domain and range of f^{-1}.
Solution:

Domain: $\{x : x \geq 0, x \in \mathbb{R}\}$
Range: $\{y : y \leq 0, y \in \mathbb{R}\}$

c) Find the equation of f.
Solution:

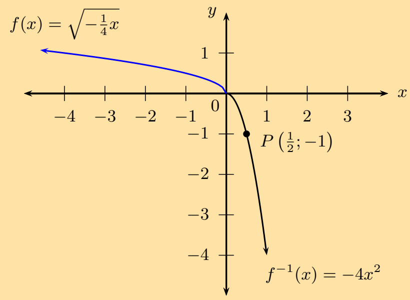
$$\begin{aligned} f^{-1}: \quad y &= -4x^2 \quad (x \geq 0) \\ f: \quad x &= -4y^2 \quad (y \geq 0) \\ -\frac{1}{4}x &= y^2 \\ \therefore y &= \sqrt{-\frac{1}{4}x} \quad (x \leq 0) \end{aligned}$$

d) State the domain and range of f.
Solution:

Domain: $\{x : x \leq 0, x \in \mathbb{R}\}$
Range: $\{y : y \geq 0, y \in \mathbb{R}\}$

e) Sketch the graphs of f and f^{-1} on the same system of axes.

Solution:



f) Is f an increasing or decreasing function?

Solution:

Decreasing function: as the value of x increases, the function value decreases.

3. Given: $g(x) = \frac{5}{2}x^2, \ \ x \geq 0$.

a) Find $g^{-1}(x)$.

Solution:

$$\begin{aligned}
\text{Let } y &= \frac{5}{2}x^2 \qquad (x \geq 0) \\
\text{Interchange } x \text{ and } y: \quad x &= \frac{5}{2}y^2 \qquad (y \geq 0) \\
\frac{2}{5}x &= y^2 \\
y &= \sqrt{\frac{2}{5}x} \qquad (x \geq 0, y \geq 0) \\
\therefore g^{-1}(x) &= \sqrt{\frac{2}{5}x} \qquad (x \geq 0)
\end{aligned}$$

b) Calculate the point(s) where g and g^{-1} intersect.

Solution:

$$\frac{5}{2}x^2 = \sqrt{\frac{2}{5}x}$$

$$\left(\frac{5}{2}x^2\right)^2 = \left(\sqrt{\frac{2}{5}x}\right)^2$$

$$\frac{25}{4}x^4 = \frac{2}{5}x$$

$$\frac{25}{4}x^4 - \frac{2}{5}x = 0$$

$$125x^4 - 8x = 0$$

$$x(125x^3 - 8) = 0$$

$$x(5x-2)(25x^2+10x+4) = 0$$

$$\therefore x = 0 \quad \text{or} \quad 5x-2=0 \quad \text{or} \quad 25x^2+10x+4=0$$

$$\text{If } x = 0, \quad y = 0$$

$$\text{If } x = \frac{2}{5}, \quad y = \frac{5}{2}\left(\frac{2}{5}\right)^2$$

$$\therefore y = \frac{2}{5}$$

$$\text{If } 25x^2+10x+4 = 0$$

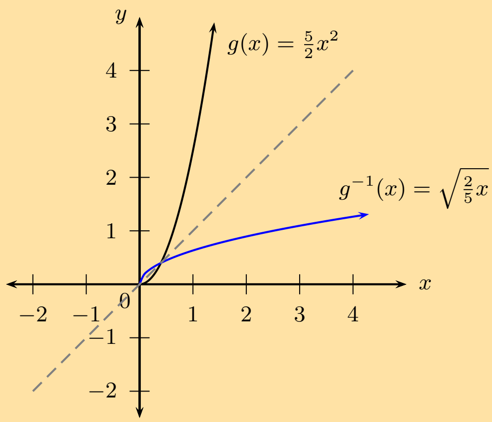
$$\text{Use quadratic formula:} \quad x = \frac{-10 \pm \sqrt{100 - 4(25)(4)}}{2(25)}$$

$$= \frac{-10 \pm \sqrt{-300}}{50}$$

$$\therefore \text{no real solution}$$

Therefore, the points of intersection are $(0;0)$ and $\left(\frac{2}{5};\frac{2}{5}\right)$.

c) Sketch g and g^{-1} on the same set of axes.
Solution:



d) Use the sketch to determine if g and g^{-1} are increasing or decreasing functions.
Solution:

$$g: \text{ as } x \text{ increases, } y \text{ also increases, } \therefore g \text{ is an increasing function.}$$

$$g^{-1}: \text{ as } x \text{ increases, } y \text{ also increases, } \therefore g^{-1} \text{ is an increasing function.}$$

e) Calculate the average gradient of g^{-1} between the two points of intersection.
Solution:

$$\text{Average gradient: } = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{\frac{2}{5} - 0}{\frac{2}{5} - 0}$$

$$= 1$$

Check answers online with the exercise code below or click on 'show me the answer'.
1. 28BN 2. 28BP 3. 28BQ

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2.6 Exponential functions

Inverse of the function $y = b^x$

Exercise 2 – 6: Finding the inverse of $y = b^x$

1. Write the following in logarithmic form:

   a) $16 = 2^4$
   Solution:
   $4 = \log_2 16$

   b) $3^{-5} = \frac{1}{243}$
   Solution:
   $-5 = \log_3 \left(\frac{1}{243}\right)$

   c) $(1{,}7)^3 = 4{,}913$
   Solution:
   $3 = \log_{1{,}7}(4{,}913)$

   d) $y = 2^x$
   Solution:
   $x = \log_2 y$

   e) $q = 4^5$
   Solution:
   $\log_4 q = 5.$

   f) $4 = y^g$
   Solution:
   $\log_y 4 = g.$

   g) $9 = (x-4)^p$
   Solution:
   $\log_{(x-4)} 9 = p.$

   h) $3 = m^{(a+4)}$
   Solution:
   $\log_m 3 = a + 4.$

2. Express each of the following logarithms in words and then write in exponential form:

   a) $\log_2 32 = 5$
   Solution:
   The logarithm of 32 to base 2 is equal to 5.
   $2^5 = 32$

   b) $\log \frac{1}{1000} = -3$
   Solution:
   The logarithm of $\frac{1}{1000}$ to base 10 is equal to -3.
   $10^{-3} = \frac{1}{1000}$

   c) $\log 0{,}1 = -1$

Solution:

The logarithm of $0{,}1$ to base 10 is equal to -1.

$10^{-1} = 0{,}1$

d) $\log_d c = b$

Solution:

The logarithm of c to base d is equal to b.

$d^b = c$

e) $\log_5 1 = 0$

Solution:

The logarithm of 1 to base 5 is equal to 0.

$5^0 = 1$

f) $\log_3 \frac{1}{81} = -4$

Solution:

The logarithm of $\frac{1}{81}$ to base 3 is equal to -4.

$3^{-4} = \frac{1}{81}$

g) $\log 100$

Solution:

$$\begin{aligned} \text{Let } \log 100 &= m \\ 10^m &= 100 \\ &= 10^2 \\ \therefore m &= 2 \\ \therefore \log 100 &= 2 \end{aligned}$$

The logarithm of 100 to base 10 is 2.

h) $\log_{\frac{1}{2}} 16$

Solution:

$$\begin{aligned} \text{Let } \log_{\frac{1}{2}} 16 &= y \\ \therefore \left(\frac{1}{2}\right)^y &= 16 \\ \therefore 2^{-y} &= 2^4 \\ -y &= 4 \\ \therefore y &= -4 \end{aligned}$$

The logarithm of 16 to base $\frac{1}{2}$ is -4.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28BS	1b. 28BT	1c. 28BV	1d. 28BW	1e. 28BX	1f. 28BY
1g. 28BZ	1h. 28C2	2a. 28C3	2b. 28C4	2c. 28C5	2d. 28C6
2e. 28C7	2f. 28C8	2g. 28C9	2h. 28CB		

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Exercise 2 – 7: Applying the logarithmic law: $\log_a x^b = b\log_a x$

Simplify the following:

1. $\log_8 10^{10}$

   Solution:

$$\log_8 10^{10} = 10\log_8 10$$

2. $\log_{16} x^y$

   Solution:

$$\log_{16} x^y = y\log_{16} x$$

3. $\log_3 \sqrt{5}$

   Solution:

$$\log_3 \sqrt{5} = \frac{\log_3 5}{2}$$

4. $\log_z y^z$

   Solution:

$$\log_z y^z = z\log_z y$$

5. $\log_y \sqrt[x]{y}$

   Solution:

$$\begin{aligned} \log_y \sqrt[x]{y} &= \log_y y^{\frac{1}{x}} \\ &= \frac{1}{x}\log_y y \\ &= \frac{1}{x} \end{aligned}$$

6. $\log_p p^q$

   Solution:

$$\begin{aligned} \log_p p^q &= q\log_p p \\ &= q(1) \\ &= q \end{aligned}$$

7. $\log_2 \sqrt[4]{8}$

   Solution:

$$\begin{aligned} \log_2 \sqrt[4]{8} &= \log_2 8^{\frac{1}{4}} \\ &= \frac{1}{4}\log_2 2^3 \\ &= \frac{1}{4}\times 3\log_2 2 \\ &= \frac{3}{4} \end{aligned}$$

8. $\log_5 \frac{1}{5}$

Solution:

$$\begin{aligned}\log_5 \frac{1}{5} &= \log_5 5^{-1}\\ &= (-1)\log_5 5\\ &= (-1)(1)\\ &= -1\end{aligned}$$

9. $\log_2 8^5$

Solution:

$$\begin{aligned}\log_2 8^5 &= 5\log_2 8\\ &= 5\log_2 2^3\\ &= 5\times 3\log_2 2\\ &= 15\end{aligned}$$

10. $\log_4 16 \times \log_3 81$

Solution:

$$\begin{aligned}\log_4 16 \times \log_3 81 &= \log_4 4^2 \times \log_3 3^4\\ &= (2)\log_4 4 \times (4)\log_3 3\\ &= (2)(1)\times(4)(1)\\ &= 8\end{aligned}$$

11. $(\log_5 25)^2$

Solution:

$$\begin{aligned}(\log_5 25)^2 &= \left(\log_5 5^2\right)^2\\ &= (2\log_5 5)^2\\ &= (2(1))^2\\ &= 4\end{aligned}$$

Alternative method:

$$\begin{aligned}(\log_5 25)^2 &= \log_5 \left(5^2\right)^2\\ &= \log_5 5^4\\ &= 4(1)\\ &= 4\end{aligned}$$

12. $\log_2 0{,}125$

Solution:

$$\begin{aligned}\log_2 0{,}125 &= \log_2 \frac{125}{1000}\\ &= \log_2 \frac{1}{8}\\ &= \log_2 8^{-1}\\ &= (-1)\log_2 2^3\\ &= (-1)(3)\log_2 2\\ &= (-3)(1)\\ &= -3\end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28CD 2. 28CF 3. 28CG 4. 28CH 5. 28CJ 6. 28CK
7. 28CM 8. 28CN 9. 28CP 10. 28CQ 11. 28CR 12. 28CS

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Exercise 2 – 8: Applying the logarithmic law: $\log_a x = \frac{\log_b x}{\log_b a}$

1. Convert the following:

   a) $\log_2 4$ to base 8

   Solution:

   $\log_2 4 = \frac{\log_8 4}{\log_8 2}$

   b) $\log_{10} 14$ to base 2

   Solution:

   $\log_{10} 14 = \frac{\log_2 14}{\log_2 10}$

   c) $\log 4\frac{1}{2}$ to base 2

   Solution:

$$\begin{aligned}\log 4\frac{1}{2} &= \log \frac{9}{2} \\ &= \frac{\log_2 \frac{9}{2}}{\log_2 10} \\ &= \frac{\log_2 9 - \log_2 2}{\log_2 10} \\ &= \frac{\log_2 9 - 1}{\log_2 10}\end{aligned}$$

   d) $\log_2 8$ to base 8

   Solution:

$$\begin{aligned}\log_2 8 &= \frac{\log_8 8}{\log_8 2} \\ &= \frac{1}{\log_8 2}\end{aligned}$$

   e) $\log_y x$ to base x

   Solution:

$$\begin{aligned}\log_y x &= \frac{\log_x x}{\log_x y} \\ &= \frac{1}{\log_x y}\end{aligned}$$

   $\therefore$ a logarithm is equal to the reciprocal of its inverse.

   f) $\log_{10} 2x$ to base 2

   Solution:

$$\begin{aligned}\log_{10} 2x &= \frac{\log_2 2x}{\log_2 10} \\ &= \frac{\log_2 2 + \log_2 x}{\log_2 10} \\ &= \frac{1 + \log_2 x}{\log_2 10}\end{aligned}$$

2. Simplify the following using a change of base:

a) $\log_2 10 \times \log_{10} 2$

Solution:

$$\begin{aligned} \log_2 10 \times \log_{10} 2 &= \frac{\log 10}{\log 2} \times \frac{\log 2}{\log 10} \\ &= 1 \end{aligned}$$

b) $\log_5 100$

Solution:

$$\begin{aligned} \log_5 100 &= \frac{\log 100}{\log 5} \\ &= \frac{\log 10^2}{\log 5} \\ &= \frac{2 \log 10}{\log 5} \\ &= \frac{2}{\log 5} \end{aligned}$$

3. If $\log 3 = 0{,}477$ and $\log 2 = 0{,}301$, determine (correct to 2 decimal places):

a) $\log_2 3$

Solution:

$$\begin{aligned} \log_2 3 &= \frac{\log 3}{\log 2} \\ &= \frac{0{,}477}{0{,}301} \\ &= 1{,}58 \end{aligned}$$

b) $\log_3 2000$

Solution:

$$\begin{aligned} \log_3 2000 &= \frac{\log 2000}{\log 3} \\ &= \frac{\log (2 \times 1000)}{\log 3} \\ &= \frac{\log 2 + \log 10^3}{\log 3} \\ &= \frac{0{,}301 + 3(1)}{0{,}477} \\ &= \frac{3{,}301}{0{,}477} \\ &= 6{,}92 \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28CV 1b. 28CW 1c. 28CX 1d. 28CY 1e. 28CZ 1f. 28D2
2a. 28D3 2b. 28D4 3a. 28D5 3b. 28D6

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Exercise 2 – 9: Logarithms using a calculator

1. Calculate the following (correct to three decimal places):

 a) $\log 3$
 Solution: $0{,}477$
 b) $\log 30$
 Solution: $1{,}477$
 c) $\log 300$
 Solution: $2{,}477$
 d) $\log 0.66$
 Solution: $-0{,}180$
 e) $\log \frac{1}{4}$
 Solution: $-0{,}602$
 f) $\log 852$
 Solution: $2{,}930$
 g) $\log(-6)$
 Solution: no value
 h) $\log_3 4$
 Solution: $1{,}262$
 i) $\log 0{,}01$
 Solution: -2
 j) $\log_2 15$
 Solution: $3{,}907$
 k) $\log_4 10$
 Solution: $1{,}661$
 l) $\log_{\frac{1}{2}} 6$
 Solution: $-2{,}585$

2. Use a calculator to determine the value of x (correct to two decimal places). Check your answer by changing to exponential form.

 a) $\log x = 0{,}6$
 Solution:

$$x = 3{,}98$$
$$\text{Check exponential form: } 10^{0{,}6} = 3{,}98$$

 b) $\log x = -2$
 Solution:

$$x = 0{,}01$$
$$\text{Check exponential form: } 10^{-2} = 0{,}01$$

 c) $\log x = 1{,}8$
 Solution:

$$x = 63{,}10$$
$$\text{Check exponential form: } 10^{1{,}8} = 63{,}10$$

d) $\log x = 5$

Solution:

$$x = 100\,000$$

Check exponential form: $10^5 = 100\,000$

e) $\log x = -0{,}5$

Solution:

$$x = 0{,}32$$

Check exponential form: $10^{-0{,}5} = 0{,}32$

f) $\log x = 0{,}076$

Solution:

$$x = 1{,}19$$

Check exponential form: $10^{0{,}076} = 1{,}19$

g) $\log x = \frac{2}{5}$

Solution:

$$x = 2{,}51$$

Check exponential form: $10^{\frac{2}{5}} = 2{,}51$

h) $\log x = -\frac{6}{5}$

Solution:

$$x = 0{,}06$$

Check exponential form: $10^{\left(-\frac{6}{5}\right)} = 0{,}06$

i) $\log_2 x = 0{,}25$

Solution:

$$\begin{aligned} \log_2 x &= 0{,}25 \\ \therefore \frac{\log x}{\log 2} &= 0{,}25 \\ \therefore \log x &= 0{,}25 \times \log 2 \\ \therefore x &= 1{,}19 \end{aligned}$$

Check exponential form: $2^{0{,}25} = 1{,}19$

j) $\log_5 x = -0{,}1$

Solution:

$$\begin{aligned} \log_5 x &= -0{,}1 \\ \therefore \frac{\log x}{\log 5} &= -0{,}1 \\ \therefore \log x &= -0{,}1 \times \log 5 \\ \therefore x &= 0{,}85 \end{aligned}$$

Check exponential form: $5^{(-0{,}10)} = 0{,}85$

k) $\log_{\frac{1}{4}} x = 2$

Solution:

$$\log_{\frac{1}{4}} x = 2$$
$$\therefore \frac{\log x}{\log \frac{1}{4}} = 2$$
$$\therefore \log x = 2 \times \log \frac{1}{4}$$
$$\therefore x = 0{,}06$$
$$\text{Check exponential form: } \left(\frac{1}{4}\right)^2 = 0{,}06$$

l) $\log_7 x = 0{,}3$

Solution:

$$\log_7 x = 0{,}3$$
$$\therefore \frac{\log x}{\log 7} = 0{,}3$$
$$\therefore \log x = 0{,}3 \times \log 7$$
$$\therefore x = 1{,}79$$
$$\text{Check exponential form: } 7^{0{,}3} = 1{,}79$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28D7	1b. 28D8	1c. 28D9	1d. 28DB	1e. 28DC	1f. 28DD
1g. 28DF	1h. 28DG	1i. 28DH	1j. 28DJ	1k. 28DK	1l. 28DM
2a. 28DN	2b. 28DP	2c. 28DQ	2d. 28DR	2e. 28DS	2f. 28DT
2g. 28DV	2h. 28DW	2i. 28DX	2j. 28DY	2k. 28DZ	2l. 28F2

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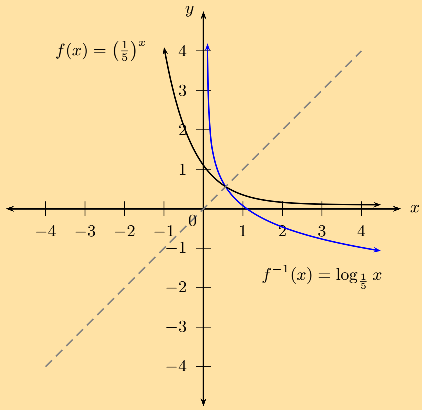
Exponential and logarithmic graphs

Exercise 2 – 10: Graphs and inverses of $y = \log_b x$

1. Given $f(x) = \left(\frac{1}{5}\right)^x$.

   a) Sketch the graphs of f and f^{-1} on the same system of axes. Label both graphs clearly.

   Solution:



b) State the intercept(s) for each graph.

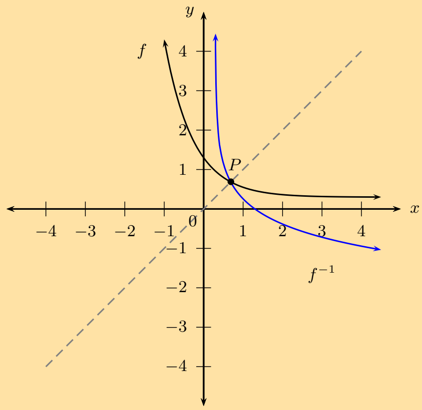
Solution:

$f:(0;1)$ and $f^{-1}:(1;0)$

c) Label P, the point of intersection of f and f^{-1}.

Solution:

Notice that the function and its inverse intersect at a point that lies on the line $y = x$.

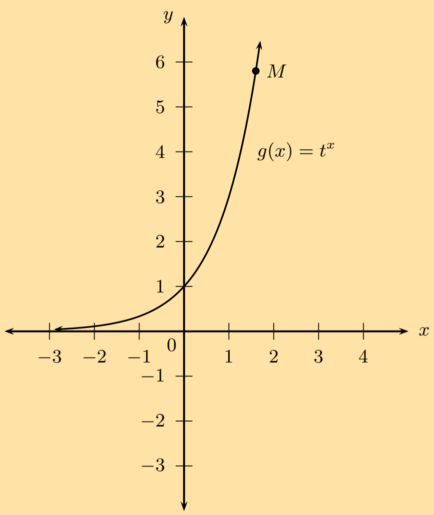


d) State the domain, range and asymptote(s) of each function.

Solution:

Function	Domain	Range	Asymptote
$f(x) = \left(\frac{1}{5}\right)^x$	$\{x : x \in \mathbb{R}\}$	$\{y : y > 0, y \in \mathbb{R}\}$	x-axis, $y = 0$
$f^{-1}(x) = \log_{\frac{1}{5}} x$	$\{x : x > 0, x \in \mathbb{R}\}$	$\{y : y \in \mathbb{R}\}$	y-axis, $x = 0$

2. Given $g(x) = t^x$ with $M\left(1\frac{3}{5}; 5\frac{4}{5}\right)$ a point on the graph of g.



a) Determine the value of t

Solution:

$$\begin{aligned}\text{Let } y &= t^x \\ 5{,}8 &= t^{1{,}6} \\ \therefore t &= \sqrt[1{,}6]{5{,}8} \\ &= 3{,}000\ldots\end{aligned}$$

$$\therefore g(x) = 3^x$$

b) Find the inverse of g.

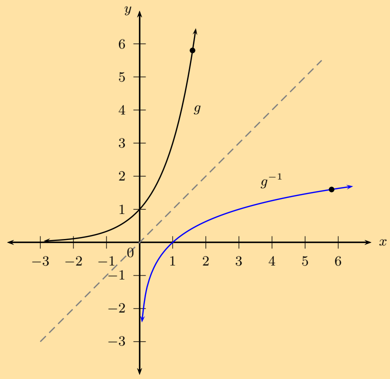
Solution:

$$\begin{aligned} \text{Let } y &= 3^x & (y > 0) \\ \text{Interchange } x \text{ and } y: \quad x &= 3^y & (x > 0) \\ \frac{x}{2} &= y^2 & \\ y &= \log_3 x & \end{aligned}$$

$$\therefore g^{-1}(x) = \log_3 x \qquad (x > 0)$$

c) Use symmetry about the line $y = x$ to sketch the graphs of g and g^{-1} on the same system of axes.

Solution:



d) Point N lies on the graph of g^{-1} and is symmetrical to point M about the line $y = x$. Determine the coordinates of N.

Solution:

$N(5{,}8; 1{,}6)$

Check answers online with the exercise code below or click on 'show me the answer'.
1. 28F4 2. 28F5

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Applications of logarithms

Exercise 2 – 11: Applications of logarithms

1. The population of Upington grows 6% every 3 years. How long will it take to triple in size?
Give your answer in years and round to the nearest integer.
Solution:
Let the population at the start be
$$P = x$$
We want to know how many periods it will take to triple in size. This means that the final population is
$$A = 3x$$
Let n be the number of periods needed to grow to a size of A. Note that every period is 3 years long, therefore we must use $\frac{n}{3}$.

The growth rate is

$$i = 6\% = \frac{6}{100}$$

We use the formula for compound growth/interest:

$$\begin{aligned} A &= P(1+i)^n \\ 3x &= x\left(1+\frac{6}{100}\right)^{\frac{n}{3}} \\ 3 &= (1{,}06)^{\frac{n}{3}} \end{aligned}$$

We now use the definition of the logarithm to write the equation in terms of n and then evaluate with a calculator:

$$\begin{aligned} \frac{n}{3} &= \log_{(1{,}06)} 3 \\ &= \frac{\log 3}{\log 1{,}06} \\ &= 18{,}854\ldots \\ \therefore n &= 3 \times 18{,}854\ldots \\ &= 56{,}562\ldots \\ &\approx 57 \end{aligned}$$

It will take approximately 57 years for the population to triple in size.

2. An ant population of 36 ants doubles every month.
   a) Determine a formula that describes the growth of the population.
      Solution:
      Let the number of months be $M_0; M_1; M_2; \ldots$

M_0		M_1		M_2		$\ldots$
36	$\rightarrow$	72	$\rightarrow$	144	$\rightarrow$	$\ldots$
	$\times 2$		$\times 2$		$\times 2$	

$$36 \times 2^0 \quad \rightarrow \quad 36 \times 2^1 \quad \rightarrow \quad 36 \times 2^2 \quad \rightarrow \quad \ldots$$

$$\text{Growth} = 36 \times 2^n$$

where n is the number of months.

   b) Calculate how long it will take for the ant population to reach a quarter of a million ants.
      Solution:

$$\begin{aligned} \text{Growth} &= 36 \times 2^n \\ \frac{1}{4} \times 1\,000\,000 &= 36 \times 2^n \\ 250\,000 &= 36 \times 2^n \\ \frac{250\,000}{36} &= 2^n \\ \therefore n &= \log_2\left(\frac{250\,000}{36}\right) \\ n &= \frac{\log\left(\frac{250\,000}{36}\right)}{\log 2} \\ \therefore n &= \frac{\log\left(\frac{250\,000}{36}\right)}{\log 2} \\ &= 12{,}7\ldots \end{aligned}$$

There will be a quarter of a million ants after about 13 months.

Check answers online with the exercise code below or click on 'show me the answer'.
1. 28F7 2. 28F8

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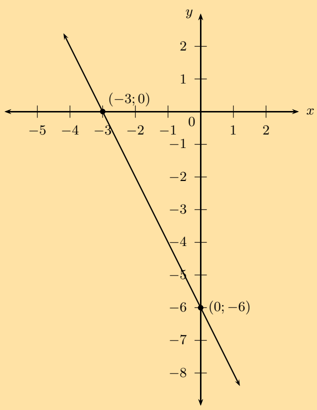
Exercise 2 – 12: End of chapter exercises

1. Given the straight line h with intercepts $(-3;0)$ and $(0;-6)$.



a) Determine the equation of h.

Solution:

$$\begin{aligned} \text{Gradient: } m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-6-0}{0-(-3)} \\ &= \frac{-6}{3} \\ &= -2 \\ \therefore h(x) &= -2x - 6 \end{aligned}$$

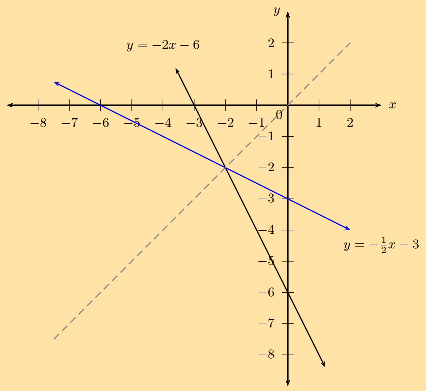
b) Find h^{-1}.

Solution:

$$\begin{aligned} \text{Let } y &= -2x - 6 \\ \text{Inverse: } x &= -2y - 6 \\ x + 6 &= -2y \\ -\frac{1}{2}(x+6) &= y \\ y &= -\frac{x}{2} - 3 \\ \therefore h^{-1}(x) &= -\frac{x}{2} - 3 \end{aligned}$$

c) Draw both graphs on the same system of axes.

Solution:



d) Calculate the coordinates of S, the point of intersection of h and h^{-1}.

Solution:

$$\begin{aligned} -\frac{x}{2} - 3 &= -2x - 6 \\ -x - 6 &= -4x - 12 \\ -x + 4x &= -12 + 6 \\ 3x &= -6 \\ \therefore x &= -2 \\ \text{If } x = -2, \quad y &= -2(-2) - 6 \\ y &= -2(-2) - 6 \\ &= -2 \end{aligned}$$

This gives the point $S(-2; -2)$

e) State the property regarding the point of intersection that will always be true for a function and its inverse.

Solution:

The value of the x-coordinate and the y-coordinate will always be the same since the point lies on the line $y = x$.

2. The inverse of a function is $f^{-1}(x) = 2x + 4$.

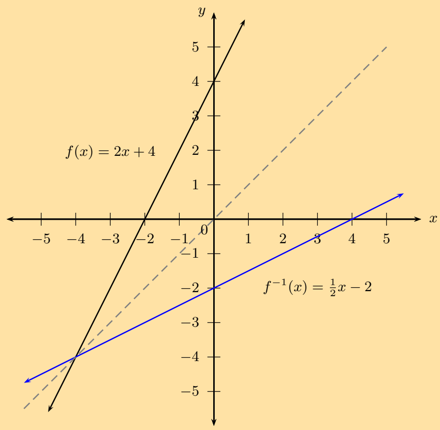
a) Determine f.

Solution:

$$\begin{aligned} \text{Let } y &= 2x + 4 \\ \text{Inverse: } x &= 2y + 4 \\ x - 4 &= 2y \\ \frac{1}{2}x - 2 &= y \\ \therefore f(x) &= \frac{1}{2}x - 2 \end{aligned}$$

b) Draw f and f^{-1} on the same set of axes. Label each graph clearly.

Solution:



c) Is f^{-1} an increasing or decreasing function? Explain your answer.

Solution:

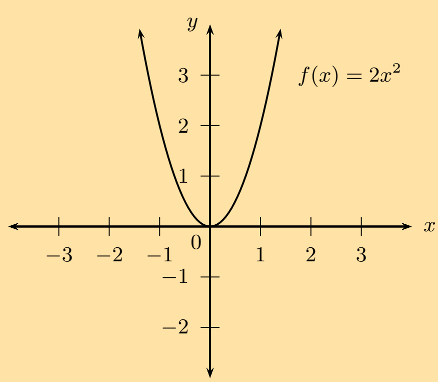
Increasing. As x increases, the function value increases. Alternative reason: gradient is positive, therefore function is increasing.

3. $f(x) = 2x^2$.

a) Draw the graph of f and state its domain and range.

Solution:

The domain is: $\{x : x \in \mathbb{R}\}$ and the range is: $\{y : y \geq 0, y \in \mathbb{R}\}$.



b) Determine the inverse and state its domain and range.

Solution:

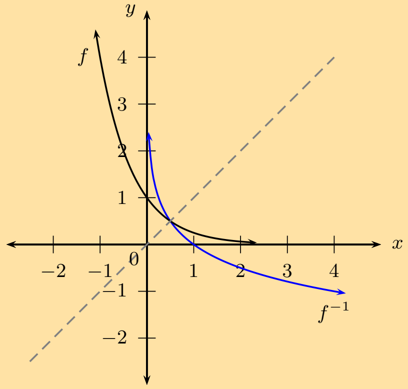
$$\text{Let } y = 2x^2$$
$$\text{Inverse: } x = 2y^2$$
$$\frac{1}{2}x = y^2$$
$$\pm\sqrt{\frac{1}{2}x} = y$$
$$\therefore y = \pm\sqrt{\frac{x}{2}} \qquad (x \geq 0)$$

Domain: $\{x : x \geq 0, x \in \mathbb{R}\}$, Range: $\{y : y \in \mathbb{R}\}$.

4. Given the function $f(x) = \left(\frac{1}{4}\right)^x$.

a) Sketch the graphs of f and f^{-1} on the same system of axes.

Solution:



b) Determine if the point $\left(-\frac{1}{2};2\right)$ lies on the graph of f.
Solution:

$$f(x)=\left(\frac{1}{4}\right)^{x}$$
$$\text{Substitute } \left(-\frac{1}{2};2\right): \quad f\left(-\frac{1}{2}\right)=\left(\frac{1}{4}\right)^{-\frac{1}{2}}$$
$$=4^{\frac{1}{2}}$$
$$=2$$

Yes, the point $\left(-\frac{1}{2};2\right)$ does lie on f.

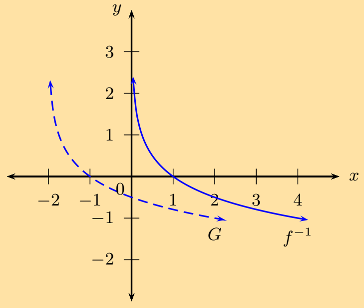
c) Write f^{-1} in the form $y=\ldots$
Solution:

$$f: \quad y=\left(\frac{1}{4}\right)^{x}$$
$$f^{-1}: \quad x=\left(\frac{1}{4}\right)^{y}$$
$$\therefore y=\log_{\frac{1}{4}} x$$
or
$$f: \quad y=(4)^{-x}$$
$$f^{-1}: \quad x=(4)^{-y}$$
$$-y=\log_4 x$$
$$\therefore y=-\log_4 x$$

$y=\log_{\frac{1}{4}} x$ or $y=-\log_4 x$

d) If the graphs of f and f^{-1} intersect at $\left(\frac{1}{2};P\right)$, determine the value of P.
Solution:
$P=\frac{1}{2}$, since the point lies on the line $y=x$.

e) Give the equation of the new graph, G, if the graph of f^{-1} is shifted 2 units to the left.
Solution:



$G(x) = -\log_4(x+2)$ or $G(x) = \log_{\frac{1}{4}}(x+2)$

f) Give the asymptote(s) of G.

Solution:

Vertical asymptote: $x = -2$

5. Consider the function $h(x) = 3^x$.

a) Write down the inverse in the form $h^{-1}(x) = \ldots$

Solution:

$$\begin{aligned} \text{Let:} \quad y &= 3^x \\ \text{Inverse:} \quad x &= 3^y \\ y &= \log_3 x \\ \therefore h^{-1}(x) &= \log_3 x \end{aligned}$$

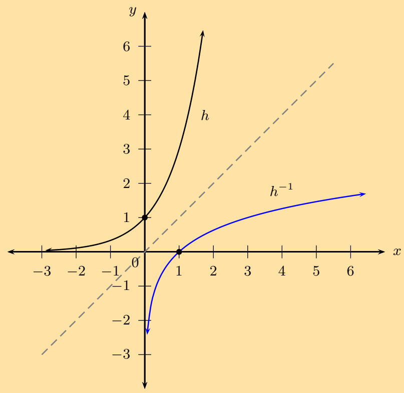
b) State the domain and range of h^{-1}.

Solution:

Domain: $\{x : x > 0, x \in \mathbb{R}\}$ and range: $\{y : y \in \mathbb{R}\}$.

c) Sketch the graphs of h and h^{-1} on the same system of axes, label all intercepts.

Solution:



d) For which values of x will $h^{-1}(x) < 0$?

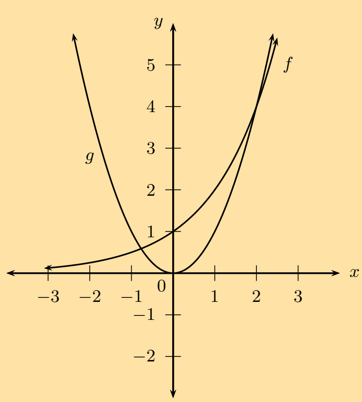
Solution:

$0 < x < 1$

6. Consider the functions $f(x) = 2^x$ and $g(x) = x^2$.

a) Sketch the graphs of f and g on the same system of axes.

Solution:



b) Determine whether or not f and g intersect at a point where $x = -1$.

Solution:

$$\begin{aligned} f(x) &= 2^x \\ f(-1) &= 2^{-1} \\ &= \frac{1}{2} \\ g(x) &= x^2 \\ g(-1) &= (-1)^2 \\ &= 1 \\ \therefore f(-1) &\neq g(-1) \end{aligned}$$

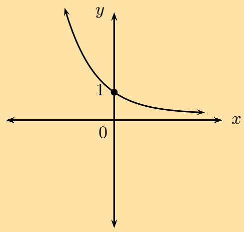
The graphs do not intersect at $x = -1$.

c) How many solutions does the equation $2^x = x^2$ have?

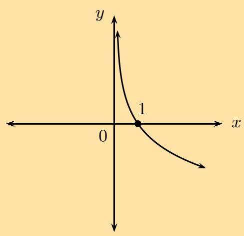
Solution:

Two

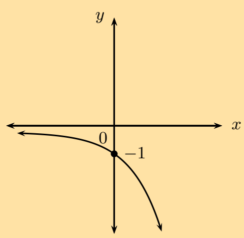
7. Below are three graphs and six equations. Write down the equation that best matches each of the graphs.



Graph 1



Graph 2



Graph 3

a) $y = \log_3 x$
b) $y = -\log_3 x$
c) $y = \log_{\frac{1}{3}} x$
d) $y = 3^x$
e) $y = 3^{-x}$
f) $y = -3^x$

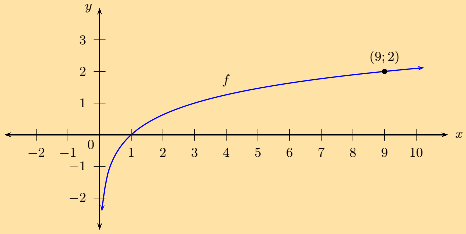
Solution:

Graph 1: $y = 3^{-x}$

Graph 2: $y = -\log_3 x$ or $y = \log_{\frac{1}{3}} x$

Graph 3: $y = -3^x$

8. Given the graph of the function $f: \quad y = \log_b x$ passing through the point $(9;2)$.



a) Show that $b = 3$.

Solution:

$$\begin{aligned} y &= \log_b x \\ 2 &= \log_b 9 \\ \therefore 9 &= b^2 \\ \therefore 3^2 &= b^2 \\ \therefore b &= 3 \end{aligned}$$

b) Determine the value of a if $(a;-1)$ lies on f.

Solution:

$$\begin{aligned} y &= \log_3 x \\ -1 &= \log_3 a \\ \therefore 3^{-1} &= a \\ \therefore a &= \frac{1}{3} \end{aligned}$$

c) Write down the new equation if f is shifted 2 units upwards.

Solution: $y = \log_3 x + 2$

d) Write down the new equation if f is shifted 1 units to the right.

Solution: $y = \log_3 (x-1)$

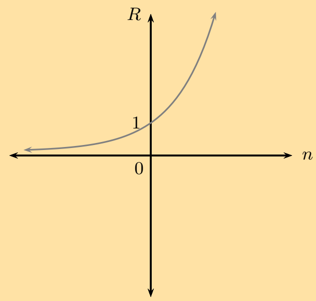
9. a) If the rhino population in South Africa starts to decrease at a rate of 7% per annum, determine how long it will take for the current rhino population to halve in size? Give your answer to the nearest integer.

Solution:

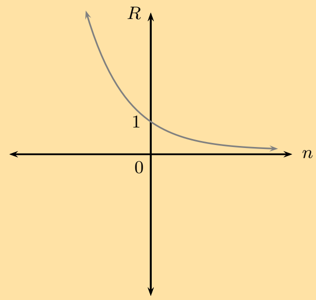
$$\begin{aligned} A &= P(1-i)^n \\ \frac{1}{2} &= \left(1 - \frac{7}{100}\right)^n \\ 0{,}5 &= (0{,}93)^n \\ \log 0{,}5 &= \log (0{,}93)^n \\ \log 0{,}5 &= n \log (0{,}93) \\ \therefore n &= \frac{\log 0{,}5}{\log (0{,}93)} \\ &= 9{,}55\ldots \end{aligned}$$

It will take less than 10 years for the current rhino population to halve in size.

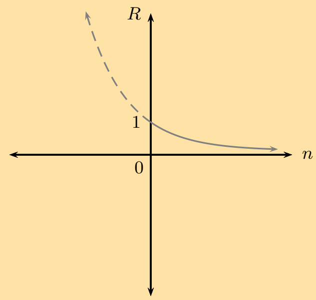
b) Which of the following graphs best illustrates the rhino population's decline? Motivate your answer.



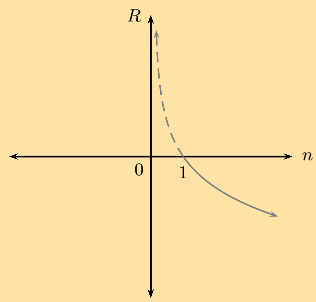
Graph A



Graph B



Graph C



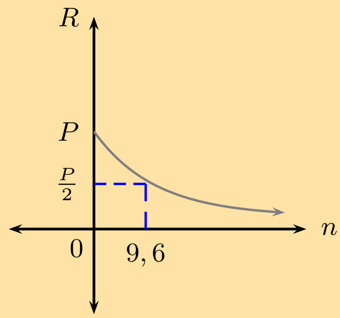
Graph D

Important note: the graphs above have been drawn as a continuous curve to show a trend. Rhino population numbers are discrete values and should be plotted points.

Solution:

Graph C

Currently $(n = 0)$ the rhino population is P. After $9{,}6$ years, it will have halved, $\frac{P}{2}$. Note: the line in the graph indicates the trend, rhino population numbers are discrete values and should be plotted points.



10. At 8 a.m. a local celebrity tweets about his new music album to 100 of his followers. Five minutes later, each of his followers retweet his message to two of their friends. Five minutes after that, each friend retweets the message to another two friends. Assume this process continues.

    a) Determine a formula that describes this retweeting process.

    Solution:

    $100 \qquad 100 \times 2 \qquad 100 \times 2^2 \qquad 100 \times 2^3$

    This is a geometric sequence: $r = 2$ and $a = 100$.

    Therefore $T_n = 100 \times 2^{n-1}$.

    b) Calculate how many retweets of the celebrity's message are sent an hour after his original tweet.

    1 hour $= 60$ minutes $= 12 \times 5$, therefore $n = 12$.

    Solution:

$$\begin{aligned} T_n &= 100 \times 2^{n-1} \\ T_{12} &= 100 \times 2^{11} \\ &= 204\ 800 \end{aligned}$$

    $204\ 800$ retweets.

    c) How long will it take for the total number of retweets to exceed 200 million?

    Solution:

$$\begin{aligned} 200 \times 10^6 &= 2 \times 10^8 \\ S_n &= \frac{a(r^n - 1)}{r - 1} \\ \therefore \frac{100(2^n - 1)}{2 - 1} &> 2 \times 10^8 \\ 2^n &> \frac{2 \times 10^8}{100} + 1 \\ 2^n &> 2\ 000\ 001 \\ n &> \log_2 2\ 000\ 001 \qquad \text{(use definition)} \\ n &> \frac{\log 2\ 000\ 001}{\log 2} \qquad \text{(change of base)} \\ n &> 20{,}9\ldots \qquad \text{5 minute periods} \end{aligned}$$

    Therefore, $\frac{21}{12} = 1{,}75$ hours.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28FB 1b. 28FC 1c. 28FD 1d. 28FF 1e. 28FG 2. 28FH
3a. 28FJ 3b. 28FK 4. 28FM 5. 28FN 6. 28FP 7. 28FQ
8. 28FR 9. 28FS 10. 28FT

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Exercise 2 – 13: Inverses (ENRICHMENT ONLY)

1. a) Given: $g(x) = -1 + \sqrt{x}$, find the inverse of $g(x)$ in the form $g^{-1}(x) = \ldots$

   Solution:

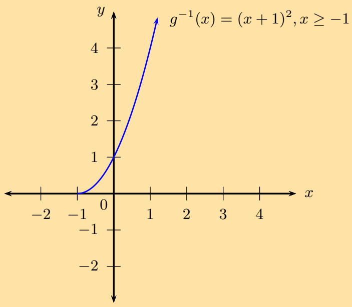
$$g(x) = -1 + \sqrt{x} \quad (x \geq 0)$$
$$\text{Let } y = -1 + \sqrt{x}$$
$$\text{Inverse: } x = -1 + \sqrt{y} \quad (y \geq 0)$$

$$\sqrt{y} = x + 1 \qquad (x \geq -1)$$
$$\therefore y = (x+1)^2$$

$$\therefore g^{-1}(x) = (x+1)^2 \qquad (x \geq -1)$$

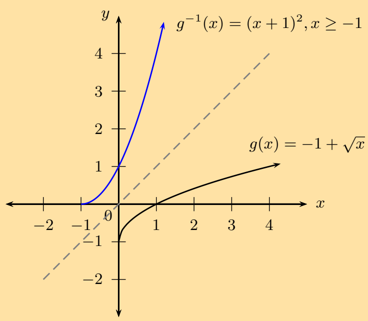
   b) Draw the graph of g^{-1}.

   Solution:



   c) Use symmetry to draw the graph of g on the same set of axes.

   Solution:



   d) Is g^{-1} a function?

   Solution:

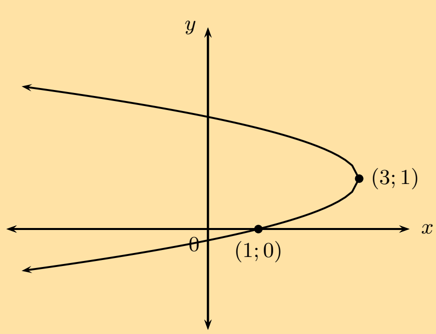
   Yes. It passes vertical line test.

   e) Give the domain and range of g^{-1}.

   Solution:

   Domain: $\{x : x \geq -1, x \in \mathbb{R}\}$, Range: $\{y : y \geq 0, y \in \mathbb{R}\}$.

2. The graph of the inverse of f is shown below:



a) Find the equation of f, given that f is a parabola of the form $y = (x+p)^2 + q$.

Solution:

First use the information provided in the graph of the inverse:

$$\text{Turning point: } (3;1)$$
$$x - \text{intercept: } (1;0)$$

To get the turning point and intercepts of the function, we invert the given coordinates. Now we can use those coordinates to find the equation of the function:

Now find the equation of the function:

$$\begin{aligned}
\text{Turning point: } (1;3) \\
x - \text{intercept: } (0;1) \\
y &= a(x-p)^2 + q \\
y &= a(x-1)^2 + 3 \\
\text{Substitute } (0;1) \quad 1 &= a(0-1)^2 + 3 \\
a &= -2 \\
\therefore y &= -2(x-1)^2 + 3
\end{aligned}$$

b) Will f have a maximum or a minimum value?

Solution:

Maximum value at $(1;3)$

c) State the domain, range and axis of symmetry of f.

Solution:

Domain: $\{x : x \in \mathbb{R}\}$ and range: $\{y : y \leq 3, y \in \mathbb{R}\}$, Axis of symmetry: $x = 1$.

3. Given: $k(x) = 2x^2 + 1$

a) If $(q;3)$ lies on k, determine the value(s) of q.

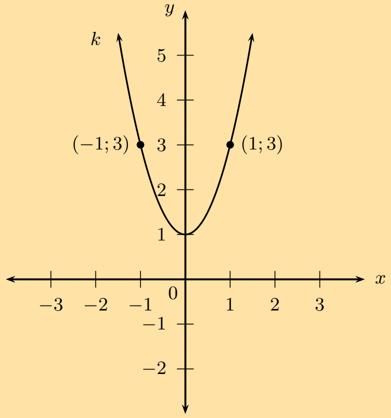
Solution:

$$\begin{aligned}
k(x) &= 2x^2 + 1 \\
\text{Substitute } (q;3) \quad 3 &= 2(q)^2 + 1 \\
2 &= 2(q)^2 \\
1 &= q^2 \\
\therefore q &= \pm 1
\end{aligned}$$

This gives the points $(-1;3)$ and $(1;3)$.

b) Sketch the graph of k, label the point(s) $(q;3)$ on the graph.

Solution:



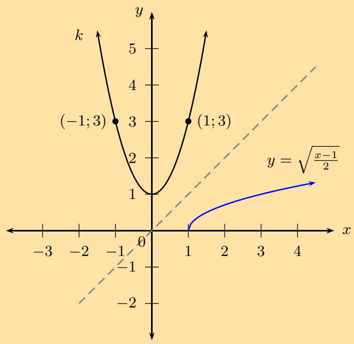
c) Find the equation of the inverse of k in the form $y = \ldots$

Solution:

$$
\begin{aligned}
k: \quad & y = 2x^2 + 1 \\
\text{Inverse:} \quad & x = 2y^2 + 1 \\
2y^2 &= x - 1 \\
y^2 &= \frac{x-1}{2} \\
y &= \pm\sqrt{\frac{x-1}{2}} \qquad (x \geq 1)
\end{aligned}
$$

d) Sketch k and $y = \sqrt{\frac{x-1}{2}}$ on the same system of axes.

Solution:

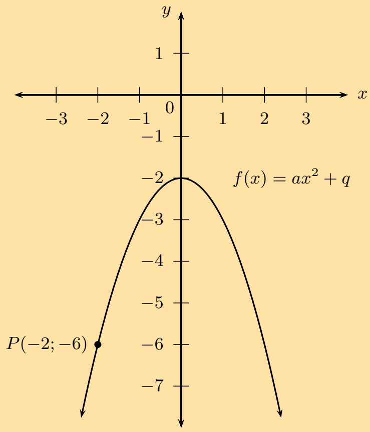


e) Determine the coordinates of the point on the graph of the inverse that is symmetrical to $(q; 3)$ about the line $y = x$.

Solution:

$(3; 1)$

4. The sketch shows the graph of a parabola $f(x) = ax^2 + q$ passing through the point $P(-2; -6)$.



a) Determine the equation of f.

Solution:

$$
\begin{aligned}
q &= -2 \\
y &= ax^2 - 2 \\
\text{Substitute } (-2; -6) \quad -6 &= a(-2)^2 - 2 \\
-6 + 2 &= 4a \\
-4 &= 4a \\
-1 &= a \\
\therefore f(x) &= -x^2 - 2
\end{aligned}
$$

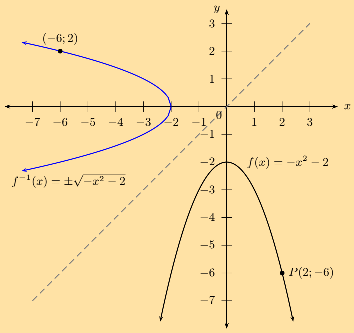
b) Determine and investigate the inverse.

Solution:

$$\begin{aligned}
\text{Let } y &= -x^2 - 2 \qquad (y \leq -2) \\
\text{Interchange } x \text{ and } y: \quad x &= -y^2 - 2 \qquad (x \leq -2) \\
x + 2 &= -y^2 \\
-x - 2 &= y^2 \\
y &= \pm\sqrt{-x-2} \qquad (x \leq -2)
\end{aligned}$$

c) Sketch the inverse and discuss the characteristics of the graph.

Solution:



The inverse is a not a function. The turning point of the inverse is $(-2;0)$ and x-intercept is $(-2;0)$.

Inverse : domain $\{x : x \leq -2, x \in \mathbb{R}\}$ range $\{y : y \in \mathbb{R}\}$

5. Given the function $H : y = x^2 - 9$.

a) Determine the algebraic formula for the inverse of H.

Solution:

$$\begin{aligned}
\text{Let } y &= x^2 - 9 \qquad (y \geq -9) \\
\text{Interchange } x \text{ and } y: \quad x &= y^2 - 9 \qquad (x \geq -9) \\
x + 9 &= y^2 \\
y &= \pm\sqrt{x+9} \qquad (x \geq -9)
\end{aligned}$$

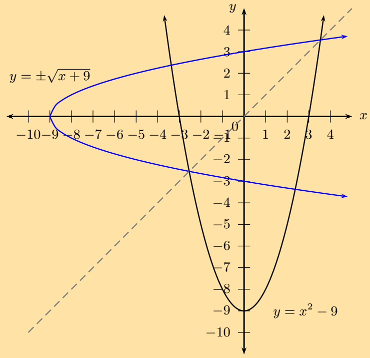
b) Draw graphs of H and its inverse on the same system of axes. Indicate intercepts and turning points.

Solution:

Determine the intercepts:

$$\begin{aligned}
\text{Let } x = 0: \quad y &= (0)^2 - 9 \\
&= -9 \\
\text{Let } y = 0: \quad 0 &= x^2 - 9 \\
x^2 &= 9 \\
\therefore x &= \pm 3
\end{aligned}$$

The intercepts are $(0;-9)$ and $(-3;0)$, $(3;0)$.



c) Is the inverse a function? Give reasons.

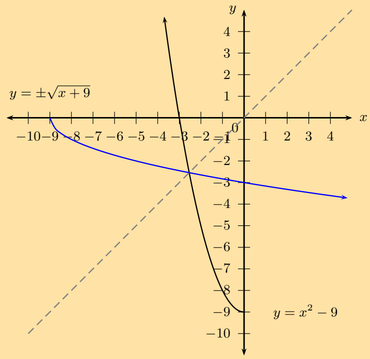
Solution:

No. The inverse does not pass the vertical line test. It is a one-to-many relation.

d) Show algebraically and graphically the effect of restricting the domain of H to $\{x : x \leq 0\}$.

Solution:

If the domain of H is restricted to $\{x : x \leq 0\}$, then the inverse is $H^{-1}(x) = -\sqrt{x^2+9} \quad (x \geq -9, y \leq 0)$.



The graph of H^{-1} cuts a vertical line only once at any one time and therefore passes the vertical line test.

Check answers online with the exercise code below or click on 'show me the answer'.
1. 28FV 2. 28FW 3. 28FX 4a. 28FY 4b. 28FZ 4c. 28G2
5. 28G3

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2.8 Enrichment: more on logarithms

Laws of logarithms

Exercise 2 – 14: Applying logarithmic law: $\log_a xy = \log_a(x) + \log_a(y)$

1. Simplify the following, if possible:

a) $\log_8(10 \times 10)$

Solution:

$$\begin{aligned}\log_8(10\times 10) &= \log_8 10 + \log_8 10\\ &= 2\log_8 10\end{aligned}$$

b) $\log_2 14$

Solution:

$$\begin{aligned}\log_2 14 &= \log_2(2\times 7)\\ &= \log_2 2 + \log_2 7\\ &= 1 + \log_2 7\end{aligned}$$

c) $\log_2(8\times 5)$

Solution:

$$\begin{aligned}\log_2(8\times 5) &= \log_2(2\times 2\times 2\times 5)\\ &= \log_2 2 + \log_2 2 + \log_2 2 + \log_2 5\\ &= 3\log_2 2 + \log_2 5\\ &= 3(1) + \log_2 5\end{aligned}$$

d) $\log_{16}(x+y)$

Solution:

$\log_{16}(x+y)$ cannot be written as separate logarithms.

e) $\log_2 2xy$

Solution:

$$\begin{aligned}\log_2 2xy &= \log_2(2\times x\times y)\\ &= \log_2 2 + \log_2 x + \log_2 y\\ &= 1 + \log_2 x + \log_2 y\end{aligned}$$

f) $\log(5+2)$

Solution:

$\log(5+2) = \log 7$

Note: $\log(5+2) \neq \log 5 + \log 2$. Do not confuse this with applying the distributive law: $a(b+c) = ab + ac$.

2. Write the following as a single term, if possible:

a) $\log 15 + \log 2$

Solution:

$$\begin{aligned}\log 15 + \log 2 &= \log(15\times 2)\\ &= \log 30\end{aligned}$$

b) $\log 1 + \log 5 + \log \frac{1}{5}$

Solution:

$$\begin{aligned}\log 1 + \log 5 + \log\frac{1}{5} &= \log\left(1\times 5\times\frac{1}{5}\right)\\ &= \log 1\\ &= 0\end{aligned}$$

c) $1 + \log_3 4$

Solution:

$$\begin{aligned} 1 + \log_3 4 &= \log_3 3 + \log_3 4 \\ &= \log_3 (3 \times 4) \\ &= \log_3 12 \end{aligned}$$

d) $(\log x)(\log y) + \log x$

Solution:

$$\begin{aligned} (\log x)(\log y) + \log x &= (\log x)[\log y + 1] \\ &= (\log x)(\log y + \log 10) \\ &= (\log x)(\log 10y) \end{aligned}$$

e) $\log 7 \times \log 2$

Solution:

$$\log 7 \times \log 2 = \log 7 \times \log 2$$

cannot be simplified further

Note: $\log 7 \times \log 2 \neq \log (7 + 2)$

f) $\log_2 7 + \log_3 2$

Solution:

This cannot be written as one term because the bases are not the same.

g) $\log_a p + \log_a q$

Solution:

$$\begin{aligned} \log_a p + \log_a q &= \log_a (p \times q) \\ &= \log_a pq \end{aligned}$$

h) $\log_a p \times \log_a q$

Solution:

This is already a single term: $(\log_a p)(\log_a q)$

3. Simplify the following:

a) $\log x + \log y + \log z$

Solution:

$$\log_x + \log y + \log z = \log xyz$$

b) $\log ab + \log bc + \log cd$

Solution:

$$\log ab + \log bc + \log cd = \log ab^2c^2d$$

c) $\log 125 + \log 2 + \log 8$

Solution:

$$\begin{aligned} \log 125 + \log 2 + \log 8 &= \log (125 \times 2 \times 8) \\ &= \log 2000 \\ &= \log (2 \times 10 \times 10 \times 10) \\ &= \log 2 + \log 10 + \log 10 + \log 10 \\ &= \log 2 + 1 + 1 + 1 \\ &= \log 2 + 3 \end{aligned}$$

d) $\log_4 \frac{3}{8} + \log_4 \frac{10}{3} + \log_4 \frac{16}{5}$

Solution:

$$\begin{aligned}\log_4 \frac{3}{8} + \log_4 \frac{10}{3} + \log_4 \frac{16}{5} &= \log_4 \left(\frac{3}{8} \times \frac{10}{3} \times \frac{16}{5}\right) \\ &= \log_4 4 \\ &= 1\end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28G4	1b. 28G5	1c. 28G6	1d. 28G7	1e. 28G8	1f. 28G9
2a. 28GB	2b. 28GC	2c. 28GD	2d. 28GF	2e. 28GG	2f. 28GH
2g. 28GJ	2h. 28GK	3a. 28GM	3b. 28GN	3c. 28GP	3d. 28GQ

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Exercise 2 – 15: Applying logarithmic law: $\log_a \frac{x}{y} = \log_a x - \log_a y$

1. Expand and simplify the following:

a) $\log \frac{100}{3}$

Solution:

$$\begin{aligned}\log \frac{100}{3} &= \log 100 - \log 3 \\ &= \log(10 \times 10) - \log 3 \\ &= \log 10 + \log 10 - \log 3 \\ &= 1 + 1 - \log 3 \\ &= 2 - \log 3\end{aligned}$$

b) $\log_2 7\frac{1}{2}$

Solution:

$$\begin{aligned}\log_2 7\frac{1}{2} &= \log_2 \frac{15}{2} \\ &= \log_2 15 - \log_2 2 \\ &= \log_2 15 - 1\end{aligned}$$

c) $\log_{16} \frac{x}{y}$

Solution:

$$\log_{16} \frac{x}{y} = \log_{16} x - \log_{16} y$$

d) $\log_{16}(x - y)$

Solution:

This cannot be simplified.

e) $\log_5 \frac{5}{8}$

Solution:

$$\begin{aligned}\log_5 \frac{5}{8} &= \log_5 5 - \log_5 8 \\ &= 1 - \log_5 8\end{aligned}$$

f) $\log_x \frac{y}{r}$

Solution:

$$\log_x \frac{y}{r} = \log_x y - \log_x r$$

2. Write the following as a single term:

 a) $\log 10 - \log 50$
 Solution:

$$\begin{aligned}\log 10 - \log 50 &= \log \frac{10}{50} \\ &= \log \frac{1}{5} \\ &= \log 5^{-1} \\ &= -\log 5\end{aligned}$$

 b) $\log_3 36 - \log_3 4$
 Solution:

$$\begin{aligned}\log_3 36 - \log_3 4 &= \log_3 \frac{36}{4} \\ &= \log_3 9 \\ &= \log_3 (3 \times 3) \\ &= \log_3 3 + \log_3 3 \\ &= 1 + 1 \\ &= 2\end{aligned}$$

 c) $\log_a p - \log_a q$
 Solution:

$$\log_a p - \log_a q = \log_a \frac{p}{q}$$

 d) $\log_a (p - q)$
 Solution:
 This cannot be simplified.

 e) $\log 15 - \log_2 5$
 Solution:
 This cannot be simplified because the bases are not the same.

 f) $\log 15 - \log 5$
 Solution:

$$\begin{aligned}\log 15 - \log 5 &= \log \frac{15}{5} \\ &= \log 3\end{aligned}$$

3. Simplify the following:

 a) $\log 450 - \log 9 - \log 5$
 Solution:

$$\begin{aligned}\log 450 - \log 9 - \log 5 &= \log \left(\frac{450}{9 \times 5}\right) \\ &= \log 10 \\ &= 1\end{aligned}$$

Alternative method:

$$\begin{aligned}\log 450 - \log 9 - \log 5 &= \log \frac{450}{9} - \log 5 \\ &= \log 50 - \log 5 \\ &= \log \frac{50}{5} \\ &= \log 10 \\ &= 1\end{aligned}$$

b) $\log \frac{4}{5} - \log \frac{3}{25} - \log \frac{1}{15}$

Solution:

$$\begin{aligned}
\log \frac{4}{5} - \log \frac{3}{25} - \log \frac{1}{15} &= \log \left(\frac{\frac{4}{5}}{\frac{3}{25} \times \frac{1}{15}} \right) \\
&= \log \left(\frac{\frac{4}{5}}{\frac{1}{125}} \right) \\
&= \log 100 \\
&= 2
\end{aligned}$$

Alternative method:

$$\begin{aligned}
\log \frac{4}{5} - \log \frac{3}{25} - \log \frac{1}{15} &= \log \frac{4}{5} - \left(\log \frac{3}{25} + \log \frac{1}{15} \right) \\
&= \log \frac{4}{5} - \log \left(\frac{3}{25} \times \frac{1}{15} \right) \\
&= \log \frac{4}{5} - \log \left(\frac{3}{25 \times 15} \right) \\
&= \log \frac{4}{5} - \log \left(\frac{1}{125} \right) \\
&= \log \left(\frac{4}{5} \div \frac{1}{125} \right) \\
&= \log \left(\frac{4}{5} \times \frac{125}{1} \right) \\
&= \log 100 \\
&= \log 10 + \log 10 \\
&= 1 + 1 \\
&= 2
\end{aligned}$$

4. Vini and Dirk complete their mathematics homework and check each other's answers. Compare the two methods shown below and decide if they are correct or incorrect:

Question:

Simplify the following:

$$\log m - \log n - \log p - \log q$$

Vini's answer:

$$\begin{aligned}
\log m - \log n - \log p - \log q &= (\log m - \log n) - \log p - \log q \\
&= \left(\log \frac{m}{n} - \log p \right) - \log q \\
&= \log \left(\frac{m}{n} \times \frac{1}{p} \right) - \log q \\
&= \log \frac{m}{np} - \log q \\
&= \log \frac{m}{np} \times \frac{1}{q} \\
&= \log \frac{m}{npq}
\end{aligned}$$

Dirk's answer:

$$\begin{aligned}
\log m - \log n - \log p - \log q &= \log m - (\log n + \log p + \log q) \\
&= \log m - \log (n \times p \times q) \\
&= \log m - \log (npq) \\
&= \log \frac{m}{npq}
\end{aligned}$$

Solution:

Both methods are correct.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28GR	1b. 28GS	1c. 28GT	1d. 28GV	1e. 28GW	1f. 28GX
2a. 28GY	2b. 28GZ	2c. 28H2	2d. 28H3	2e. 28H4	2f. 28H5
3a. 28H6	3b. 28H7	4. 28H8			

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Simplification of logarithms

Exercise 2 – 16: Simplification of logarithms

Simplify the following without using a calculator:

1. $8^{\frac{2}{3}} + \log_2 32$

   Solution:

$$\begin{aligned} 8^{\frac{2}{3}} + \log_2 32 &= \left(2^3\right)^{\frac{2}{3}} + \log_2 \left(2^5\right) \\ &= 2^2 + 5\log_2 2 \\ &= 4 + 5 \\ &= 9 \end{aligned}$$

2. $2\log 3 + \log 2 - \log 5$

   Solution:

$$\begin{aligned} 2\log 3 + \log 2 - \log 5 &= \log 3^2 + \log 2 - \log 5 \\ &= \log \frac{9 \times 2}{5} \\ &= \log \frac{18}{5} \end{aligned}$$

3. $\log_2 8 - \log 1 + \log_4 \frac{1}{4}$

   Solution:

$$\begin{aligned} \log_2 8 - \log 1 + \log_4 \frac{1}{4} &= \log_2 2^3 - 0 + \log_4 4^{(-1)} \\ &= 3\log_2 2 - 1\log_4 4 \\ &= 3(1) - 1(1) \\ &= 2 \end{aligned}$$

4. $\log_8 1 - \log_5 \frac{1}{25} + \log_3 9$

   Solution:

$$\begin{aligned} \log_8 1 - \log_5 \frac{1}{25} + \log_3 9 &= 0 - \log_5 5^{(-2)} + \log_3 3^2 \\ &= -(-2)\log_5 5 + 2\log_3 3 \\ &= 2(1) + 2(1) \\ &= 4 \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.
1. 28H9 2. 28HB 3. 28HC 4. 28HD

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Solving logarithmic equations

Exercise 2 – 17: Solving logarithmic equations

1. Determine the value of a (correct to 2 decimal places):

a) $\log_3 a - \log 1{,}2 = 0$

Solution:

$$\log_3 a - \log 1{,}2 = 0$$
$$\log_3 a = \log 1{,}2$$
Change to exponential form:
$$3^{\log 1{,}2} = a$$
$$\therefore a = 1{,}09$$

Alternative (longer) method:

$$\log_3 a - \log 1{,}2 = 0$$
$$\log_3 a = \log 1{,}2$$
$$\frac{\log a}{\log 3} = \log 1{,}2$$
$$\log a = \log 3 \times \log 1{,}2$$
$$\log a = 0{,}037\ldots$$
$$\therefore a = 1{,}09$$

b) $\log_2 (a - 1) = 1{,}5$

Solution:

$$\log_2 (a - 1) = 1{,}5$$
Change to exponential form:
$$2^{1{,}5} = a - 1$$
$$2^{1{,}5} + 1 = a$$
$$\therefore a = 3{,}83$$

Alternative (longer) method:

$$\log_2 (a - 1) = 1{,}5$$
$$\frac{\log (a - 1)}{\log 2} = 1{,}5$$
$$\log (a - 1) = \log 2 \times 1{,}5$$
$$\therefore a - 1 = 2{,}83\ldots$$
$$\therefore a = 3{,}83$$

c) $\log_2 a - 1 = 1{,}5$

Solution:

$$\begin{aligned}\log_2 a - 1 &= 1{,}5\\ \log_2 a &= 2{,}5\\ \text{Change to exponential form:}\\ 2^{2{,}5} &= a\\ \therefore a &= 5{,}66\end{aligned}$$

Alternative (longer) method:

$$\begin{aligned}\log_2 a - 1 &= 1{,}5\\ \frac{\log a}{\log 2} &= 2{,}5\\ \log a &= \log 2 \times 2{,}5\\ \therefore a &= 5{,}66\end{aligned}$$

d) $3^a = 2{,}2$

Solution:

$$\begin{aligned}3^a &= 2{,}2\\ \therefore a &= \log_3 2{,}2\\ &= \frac{\log 2{,}2}{\log 3}\\ \therefore a &= 0{,}72\end{aligned}$$

e) $2^{(a+1)} = 0{,}7$

Solution:

$$\begin{aligned}2^{(a+1)} &= 0{,}7\\ \therefore a + 1 &= \log_2 0{,}7\\ \therefore a &= \frac{\log 0{,}7}{\log 2} - 1\\ &= -1{,}51\end{aligned}$$

f) $(1{,}03)^{\frac{a}{2}} = 2{,}65$

Solution:

$$\begin{aligned}(1{,}03)^{\frac{a}{2}} &= 2{,}65\\ \therefore \frac{a}{2} &= \log_{1{,}03} 2{,}65\\ \therefore a &= 2 \times \frac{\log 2{,}65}{\log 1{,}03}\\ &= 65{,}94\end{aligned}$$

g) $(9)^{(1-2a)} = 101$

Solution:

$$\begin{aligned}(9)^{(1-2a)} &= 101\\ \therefore 1 - 2a &= \log_9 101\\ \therefore 1 - \frac{\log 101}{\log 9} &= 2a\\ -1{,}10\ldots &= 2a\\ \therefore -0{,}55 &= a\end{aligned}$$

2. Given $y = 3^x$.

   a) Write down the equation of the inverse of $y = 3^x$ in the form $y = \ldots$

      Solution:

      $y = \log_3 x$

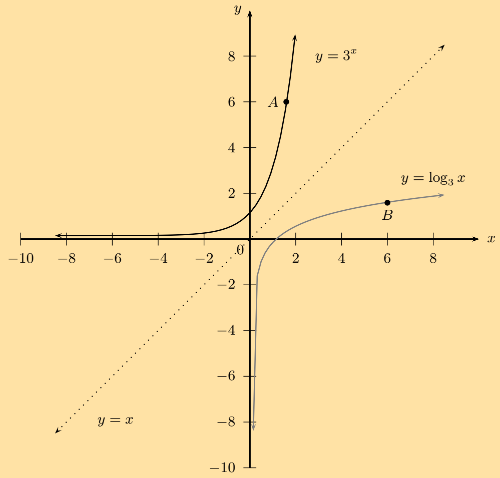
   b) If $6 = 3^p$, determine the value of p (correct to one decimal place).

      Solution:

$$\begin{aligned} p &= \log_3 6 \\ &= \frac{\log 6}{\log 3} \\ &= 1{,}6 \end{aligned}$$

   c) Draw the graph of $y = 3^x$ and its inverse. Plot the points $A(p; 6)$ and $B(6; p)$.

      Solution:



Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28HF 1b. 28HG 1c. 28HH 1d. 28HJ 1e. 28HK 1f. 28HM
1g. 28HN 2. 28HP

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Summary

Exercise 2 – 18: Logarithms (ENRICHMENT ONLY)

1. State whether the following are true or false. If false, change the statement so that it is true.

   a) $\log t + \log d = \log (t + d)$

      Solution:

      False: $\log t + \log d = \log (t \times d)$

   b) If $p^q = r$, then $q = \log_r p$

      Solution:

      False: $q = \log_p r$

   c) $\log \frac{A}{B} = \log A - \log B$

Solution:
True

d) $\log A - B = \frac{\log A}{\log B}$

Solution:
False: $\log(A) - B$ cannot be simplified further.

e) $\log_{\frac{1}{2}} x = -\log_2 x$

Solution:
True

f) $\log_k m = \frac{\log_p k}{\log_p m}$

Solution:
False: $\log_k m = \frac{\log_p m}{\log_p k}$

g) $\log_n \sqrt{b} = \frac{1}{2}\log_n b$

Solution:
True

h) $\log_p q = \frac{1}{\log_q p}$

Solution:
True

i) $2\log_2 a + 3\log a = 5\log a$

Solution:
False: bases are different

j) $5\log x + 10\log x = 5\log x^3$

Solution:
True

k) $\frac{\log_n a}{\log_n b} = \log_n \frac{a}{b}$

Solution:
False: cannot be simplified to single logarithm

l) $\log(A+B) = \log A + \log B$

Solution:
False: do not confuse with $\log(AB) = \log A + \log B$ or with the distributive law $x(a+b) = ax + ab$.

m) $\log 2a^3 = 3\log 2a$

Solution:
False: $\log 2a^3 = \log 2 + 3\log a$

n) $\frac{\log_n a}{\log_n b} = \log_n (a-b)$

Solution:
False: do not confuse with $\log_n\left(\frac{a}{b}\right) = \log_n a - \log_n b$. LHS cannot be simplified.

2. Simplify the following without using a calculator:

a) $\log 7 - \log 0{,}7$

Solution:

$$\begin{aligned}\log 7 - \log 0{,}7 &= \log \frac{7}{0{,}7} \\ &= \log 10 \\ &= 1\end{aligned}$$

b) $\log 8 \times \log 1$

Solution:

$$\begin{aligned}\log 8 \times \log 1 &= \log 8 \times 0 \\ &= 0\end{aligned}$$

c) $\log \frac{1}{3} + \log 300$

Solution:

$$\begin{aligned}\log\frac{1}{3}+\log 300 &= \log\left(\frac{1}{3}\times 300\right)\\ &= \log 100\\ &= \log 10^2\\ &= 2\log 10\\ &= 2\end{aligned}$$

d) $2\log 3+\log 2-\log 6$

Solution:

$$\begin{aligned}2\log 3+\log 2-\log 6 &= \log 3^2+\log 2-\log 6\\ &= \log\frac{9\times 2}{6}\\ &= \log\frac{18}{6}\\ &= \log 3\end{aligned}$$

3. Given $\log 5 = 0{,}7$. Find the value of the following without using a calculator:

a) $\log 50$

Solution:

$$\begin{aligned}\log 50 &= \log 5+\log 10\\ &= 0{,}7+1\\ &= 1{,}7\end{aligned}$$

b) $\log 20$

Solution:

$$\begin{aligned}\log 20 &= \log\frac{100}{5}\\ &= \log 100-\log 5\\ &= 2-0{,}7\\ &= 1{,}3\end{aligned}$$

c) $\log 25$

Solution:

$$\begin{aligned}\log 25 &= \log 5^2\\ &= 2\times\log 5\\ &= 2\times 0{,}7\\ &= 1{,}4\end{aligned}$$

d) $\log_2 5$

Solution:

$$
\begin{aligned}
\log_2 5 &= \frac{\log 5}{\log 2} \\
&= \frac{\log 5}{\log \frac{10}{5}} \\
&= \frac{\log 5}{\log 10 - \log 5} \\
&= \frac{0{,}7}{1 - 0{,}7} \\
&= \frac{0{,}7}{0{,}3} \\
&= \frac{7}{10} \times \frac{10}{3} \\
&= \frac{7}{3}
\end{aligned}
$$

e) $10^{0{,}7}$

Solution:

If $\log 5 = 0{,}7$, then $10^{0{,}7} = 5$.

4. Given $A = \log_8 1 - \log_5 \frac{1}{25} + \log_3 9$.

a) Without using a calculator, show that $A = 4$.

Solution:

$$
\begin{aligned}
A &= \log_8 1 - \log_5 \frac{1}{25} + \log_3 9 \\
&= 0 - \log_5 5^{-2} + \log_3 3^2 \\
&= 0 - (-2)\log_5 5 + (2)\log_3 3 \\
&= 2 + 2 \\
&= 4
\end{aligned}
$$

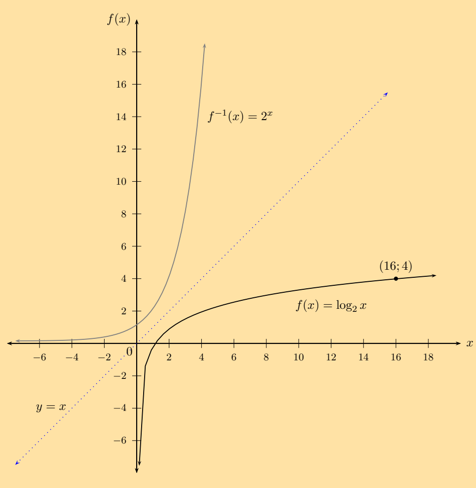
b) Now solve for x if $\log_2 x = A$.

Solution:

$$
\begin{aligned}
\log_2 x &= 4 \\
\therefore x &= 2^4 \\
x &= 16
\end{aligned}
$$

c) Let $f(x) = \log_2 x$. Draw the graph of f and f^{-1}. Indicate the point $(x; A)$ on the graph.

Solution:



5. Solve for x if $\frac{35^x}{7^x} = 15$. Give answer correct to two decimal places.

Solution:

$$\begin{aligned}
\frac{35^x}{7^x} &= 15 \\
\frac{7^x \,.\, 5^x}{7^x} &= 15 \\
5^x &= 15 \\
\therefore x &= \log_5 15 \\
&= \frac{\log 15}{\log 5} \\
&= 1{,}68
\end{aligned}$$

6. Given $f(x) = 5 \times (1{,}5)^x$ and $g(x) = \left(\frac{1}{4}\right)^x$.

   a) For which integer values of x will $f(x) < 295$.

   Solution:

$$\begin{aligned}
5 \times (1{,}5)^x &< 295 \\
\therefore \log (1{,}5)^x &< \log 59 \\
x \log (1{,}5) &< \log 59 \\
x &< \frac{\log 59}{\log (1{,}5)} \qquad \text{note: } \log (1{,}5) > 0 \\
x &< 10{,}0564\ldots
\end{aligned}$$

   Therefore, $x < 10, (x \in \mathbb{Z})$.

   b) For which values of x will $g(x) \geq 2{,}7 \times 10^{-7}$. Give answer to the nearest integer.

   Solution:

$$\begin{aligned}
\left(\frac{1}{4}\right)^x &\geq 2{,}7 \times 10^{-7} \\
\log \left(\frac{1}{4}\right)^x &\geq \log 2{,}7 \times 10^{-7} \\
x \log \left(\frac{1}{4}\right) &\geq \log 2{,}7 \times 10^{-7} \\
x &\leq \frac{\log 2{,}7 \times 10^{-7}}{\log \left(\frac{1}{4}\right)} \qquad \text{note: } \log \left(\frac{1}{4}\right) < 0 \\
&\leq 10{,}9\ldots \\
\therefore x &< 11
\end{aligned}$$

   Important: notice that the inequality sign changed direction when we divided both sides by $\log \left(\frac{1}{4}\right) = -\log 4$, since it has a negative value.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28HQ	1b. 28HR	1c. 28HS	1d. 28HT	1e. 28HV	1f. 28HW
1g. 28HX	1h. 28HY	1i. 28HZ	1j. 28J2	1k. 28J3	1l. 28J4
1m. 28J5	1n. 28J6	2a. 28J7	2b. 28J8	2c. 28J9	2d. 28JB
3a. 28JC	3b. 28JD	3c. 28JF	3d. 28JG	3e. 28JH	4. 28JJ
5. 28JK	6. 28JM				

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CHAPTER 3

Finance

- Discuss terminology.
- Very important to emphasize not rounding off calculations until final answer as this affects accuracy.
- Learners should do calculation in one step using the memory function on their calculators.
- Draw timelines showing the different time periods, interest rates and any deposits/withdrawals.
- Explain and discuss the difference between future and present value annuities.
- Learners must always check how often a given interest rate is compounded.
- Learners must be careful to calculate the correct number of payments for the term of the investment.

3.1 Calculating the period of an investment

Exercise 3 – 1: Determining the period of an investment

1. Nzuzo invests R 80 000 at an interest rate of 7,5% per annum compounded yearly. How long will it take for his investment to grow to R 100 000?

   Solution:

$$
\begin{aligned}
A &= P(1+i)^n \\
100\ 000 &= 80\ 000\left(1+\frac{7{,}5}{100}\right)^n \\
\frac{100\ 000}{80\ 000} &= (1{,}075)^n \\
\frac{5}{4} &= (1{,}075)^n \\
\therefore n &= \log_{1{,}075} 1{,}25 \\
&= \frac{\log 1{,}25}{\log 1{,}075} \\
&= 3{,}09\ldots
\end{aligned}
$$

   It will take just over 3 years.

2. Sally invests R 120 000 at an interest rate of 12% per annum compounded quarterly. How long will it take for her investment to double?

   Solution:

$$
\begin{aligned}
A &= P(1+i)^n \\
240\ 000 &= 120\ 000\left(1+\frac{12}{100\times 4}\right)^{4n} \\
\frac{240\ 000}{120\ 000} &= (1{,}03)^{4n} \\
2 &= (1{,}03)^{4n} \\
\therefore 4n &= \log_{1{,}03} 2 \\
&= \frac{\log 2}{\log 1{,}03} \\
&= 23{,}449\ldots \\
\therefore n &= 5{,}86\ldots
\end{aligned}
$$

   It will take just over 5 years and 10 months.

3. When Banele was still in high school he deposited R 2250 into a savings account with an interest rate of 6,99% per annum compounded yearly. How long ago did Banele open the account if the balance is now R 2882,53? Write the answer as a combination of years and months.

   Solution:

   This is a compound interest problem:

   $$A = P(1+i)^n$$

   Where:

   $$A = 2882{,}53$$
   $$P = 2250$$
   $$i = 0{,}0699$$

   Now substitute the known values and solve for n:

   $$\begin{aligned} 2882{,}53 &= 2250\,(1+0{,}0699)^n \\ 2882{,}53 &= 2250(1{,}0699)^n \\ \frac{2882{,}53}{2250} &= (1{,}0699)^n \end{aligned}$$

   Change to logarithmic form:

   $$\begin{aligned} n &= \log_{1{,}0699}\left(\frac{2882{,}53}{2250}\right) \\ n &= 3{,}6666\ldots \end{aligned}$$

   Banele left the money in the account for about $3{,}67$ years.

   However, we must give our answer in terms of years and months, not as a decimal number of years. $3{,}6666\ldots$ years means 3 years and some number of months; to figure out how many months, we need to convert $0{,}6666\ldots$ years into months.

   We know that there are 12 months in a year.

   To convert $0{,}6666\ldots$ years into months, do the following:

   $$(0{,}6666\ldots)\text{ year} \times \left(\frac{12\text{ months}}{\text{year}}\right) = 8\text{ months}$$

   Banele deposited the money into the account 3 years and 8 months ago.

4. The annual rate of depreciation of a vehicle is 15%. A new vehicle costs R 122 000. After how many years will the vehicle be worth less than R 40 000?

   Solution:

   $$\begin{aligned} A &= P(1-i)^n \\ 40\,000 &= 122\,000\left(1-\frac{15}{100}\right)^n \\ \frac{40\,000}{122\,000} &= (0{,}85)^n \\ \frac{20}{61} &= (0{,}85)^n \\ \therefore n &= \log_{0{,}85}\frac{20}{61} \\ &= \frac{\log\frac{20}{61}}{\log 0{,}85} \\ &= 6{,}86\ldots \end{aligned}$$

   The vehicle will be worth less than R 40 000 after about 7 years.

5. Some time ago, a man opened a savings account at KMT South Bank and deposited an amount of R 2100. The balance of his account is now R 3160,59. If the account gets 8,52% compound interest p.a., determine how many years ago the man made the deposit.

Solution:

We write the compound interest formula and the given information:

$$A = P(1+i)^n$$

Where:

$$A = 3160{,}59$$
$$P = 2100$$
$$i = 0{,}0852$$

We know everything except for the value of n. Substitute and then solve for n.

$$\begin{aligned} 3160{,}59 &= 2100\,(1+0{,}0852)^n \\ 3160{,}59 &= 2100(1{,}0852)^n \\ \frac{3160{,}59}{2100} &= (1{,}0852)^n \end{aligned}$$

Use the definition of a logarithm to solve for n:

$$n = \log_{1{,}0852}\left(\frac{3160{,}59}{2100}\right)$$

Use a calculator to evaluate the $\log$:

$$n = 4{,}999\ldots$$

The man made the deposit 5 years ago.

6. Mr. and Mrs. Dlamini want to save money for their son's university fees. They deposit R 7000 in a savings account with a fixed interest rate of $6{,}5\%$ per year compounded annually. How long will take for this deposit to double in value?

Solution:

$$A = P(1+i)^n$$

Where:

$$A = 14\ 000$$
$$P = 7000$$
$$i = 0{,}065$$

$$\begin{aligned} 14\ 000 &= 7000\left(1+\frac{6{,}5}{100}\right)^n \\ 2 &= (1{,}065)^n \\ \therefore n &= \log_{1{,}065} 2 \\ &= \frac{\log 2}{\log 1{,}065} \\ &= 11{,}00\ldots \\ &= 11 \end{aligned}$$

It will take 11 years for their deposit to double in value.

7. A university lecturer retires at the age of 60. She has saved R 300 000 over the years.

    a) She decides not to let her savings decrease at a rate faster than 15% per year. How old will she be when the value of her savings is less than R 50 000?

    Solution:

    $$A = P(1+i)^n$$

    Where:

    $$A = 50\ 000$$
    $$P = 300\ 000$$
    $$i = 0{,}15$$

$$
\begin{aligned}
50\ 000 &= 300\ 000\left(1-\frac{15}{100}\right)^{n} \\
\frac{1}{6} &= (0{,}85)^{n} \\
\therefore n &= \frac{\log \frac{1}{6}}{\log 0{,}85} \\
&= 11{,}024\ldots
\end{aligned}
$$

If she manages her money carefully, she will be 71 years or older.

b) If she doesn't use her savings and invests all her money in an investment account that earns a fixed interest rate of 5,95% per annum, how long will it take for her investment to grow to R 390 000?

Solution:

$$A = P(1+i)^{n}$$

Where:

$$
\begin{aligned}
A &= 390\ 000 \\
P &= 300\ 000 \\
i &= 0{,}0595
\end{aligned}
$$

$$
\begin{aligned}
390\ 000 &= 300\ 000\left(1+\frac{5{,}95}{100}\right)^{n} \\
1{,}3 &= (1{,}0595)^{n} \\
\therefore n &= \frac{\log 1{,}3}{\log 1{,}0595} \\
&= 4{,}539\ldots
\end{aligned}
$$

It will take less than 5 years.

8. Simosethu puts R 450 into a bank account at the Bank of Upington. Simosethu's account pays interest at a rate of 7,11% p.a. compounded monthly. After how many years will the bank account have a balance of R 619,09?

Solution:

Use the compound interest formula and determine the value of n.

$$A = P(1+i)^{n}$$

Where:

$$
\begin{aligned}
A &= 619{,}09 \\
P &= 450 \\
i &= 0{,}0711
\end{aligned}
$$

In this question the interest is payable every month. Therefore $i \rightarrow \frac{0{,}0711}{12}$ and $n \rightarrow (n \times 12)$. In this case, n represents the number of years; the product $(n \times 12)$ represents the number of times the bank pays interest into the account.

$$619{,}09 = 450\left(1+\frac{0{,}0711}{12}\right)^{(n\times 12)}$$

$$
\begin{aligned}
619{,}09 &= 450(1{,}00592\ldots)^{12n} \\
\frac{619{,}09}{450} &= (1{,}00592\ldots)^{12n} \\
1{,}37575\ldots &= (1{,}00592\ldots)^{12n}
\end{aligned}
$$

At this point we must change the equation to logarithmic form:

$$\begin{aligned} 12n &= \log_{1,005925}(1,37575\ldots) \\ 12n &= 54 \end{aligned}$$

To find the number of years, we solve for n:

$$\begin{aligned} 12n &= 54 \\ n &= \frac{54}{12} \\ n &= 4,5 \end{aligned}$$

The money has been in the Simosethu's account for $4,5$ years.

Check answers online with the exercise code below or click on 'show me the answer'.
1. 28JP 2. 28JQ 3. 28JR 4. 28JS 5. 28JT 6. 28JV
7. 28JW 8. 28JX

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3.3 Future value annuities

Exercise 3 – 2: Future value annuities

1. Shelly decides to start saving money for her son's future. At the end of each month she deposits R 500 into an account at Durban Trust Bank, which earns an interest rate of $5,96\%$ per annum compounded quarterly.

   a) Determine the balance of Shelly's account after 35 years.

   Solution:

   Write down the given information and the future value formula:

   $$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

   $$\begin{aligned} x &= 500 \\ i &= \frac{0,0596}{4} \\ n &= 35 \times 4 = 140 \end{aligned}$$

   Substitute the known values and use a calculator to determine F:

   $$\begin{aligned} F &= \frac{500\left[(1+\frac{0,0596}{4})^{140} - 1\right]}{\frac{0,0596}{4}} \\ &= \text{R } 232\ 539,41 \end{aligned}$$

   b) How much money did Shelly deposit into her account over the 35 year period?

   Solution:

   Calculate the total value of deposits into the account:

   Shelly deposited R 500 each month for 35 years:

   $$\begin{aligned} \text{Total deposits: } &= \text{R } 500 \times 12 \times 35 \\ &= \text{R } 210\ 000 \end{aligned}$$

c) Calculate how much interest she earned over the 35 year period.

Solution:

Calculate the total interest earned:

$$\begin{aligned}\text{Total interest} &= \text{final account balance} - \text{total value of all deposits}\\ &= \text{R } 232\,539{,}41 - \text{R } 210\,000\\ &= \text{R } 22\,539{,}41\end{aligned}$$

2. Gerald wants to buy a new guitar worth R 7400 in a year's time. How much must he deposit at the end of each month into his savings account, which earns a interest rate of 9,5% p.a. compounded monthly?

Solution:

Write down the given information and the future value formula:

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

To determine the monthly payment amount, we make x the subject of the formula:

$$x = \frac{F \times i}{\left[(1+i)^n - 1\right]}$$

$$\begin{aligned}F &= 7400\\ i &= \frac{0{,}095}{12}\\ n &= 1 \times 12 = 12\end{aligned}$$

Substitute the known values and calculate x:

$$\begin{aligned}x &= \frac{7400 \times \frac{0{,}095}{12}}{\left[(1+\frac{0{,}095}{12})^{12} - 1\right]}\\ &= \text{R } 590{,}27\end{aligned}$$

Write the final answer:

Gerald must deposit R 590,27 each month so that he can afford his guitar.

3. A young woman named Grace has just started a new job, and wants to save money for the future. She decides to deposit R 1100 into a savings account every month. Her money goes into an account at First Mutual Bank, and the account earns 8,9% interest p.a. compounded every month.

a) How much money will Grace have in her account after 29 years?

Solution:

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

Where: $x = \text{R } 1100$

$i = 0{,}089$

$n = 29$

$$\begin{aligned}F &= \frac{(1100)\left[\left(1+\frac{0{,}089}{12}\right)^{(29\times 12)} - 1\right]}{\left(\frac{0{,}089}{12}\right)}\\ &= \text{R } 1\,792\,400{,}11\end{aligned}$$

After 29 years, Grace will have R 1 792 400,11 in her account.

b) How much money did Grace deposit into her account by the end of the 29 year period?

Solution:

The total amount of money Grace saves each **year** is $1100 \times 12 =$ R 13 200. From that we can determine the total amount she saves by multiplying by the number of years: $13\ 200 \times 29 =$ R 382 800.

After 29 years, Grace deposited a total of R 382 800 into her account.

4. Ruth decides to save for her retirement so she opens a savings account and immediately deposits R 450 into the account. Her savings account earns 12% per annum compounded monthly. She then deposits R 450 at the end of each month for 35 years. What is the value of her retirement savings at the end of the 35 year period?

Solution:

Write down the given information and the future value formula:

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$\begin{aligned} x &= 450 \\ i &= \frac{0{,}12}{12} \\ n &= 1 + (35 \times 12) = 421 \end{aligned}$$

Substitute the known values and calculate F:

$$\begin{aligned} F &= \frac{450\left[(1 + \frac{0{,}12}{12})^{421} - 1\right]}{\frac{0{,}12}{12}} \\ &= \text{R } 2\ 923\ 321{,}08 \end{aligned}$$

Write the final answer:

Ruth will have saved R 2 923 321,08 for her retirement.

5. Musina MoneyLenders offer a savings account with an interest rate of 6,13% p.a. compounded monthly. Monique wants to save money so that she can buy a house when she retires. She decides to open an account and make regular monthly deposits. Her goal is to end up with R 750 000 in her account after 35 years.

a) How much must Monique deposit into her account each month in order to reach her goal?

Solution:

$$\begin{aligned} F &= \frac{x\left[(1+i)^n - 1\right]}{i} \\ F &= \text{R } 750\ 000 \\ i &= 0{,}0613 \\ n &= 35 \end{aligned}$$

$$\begin{aligned} 750\ 000 &= \frac{x\left[\left(1 + \frac{0{,}0613}{12}\right)^{(35\times 12)}\right]}{\left(\frac{0{,}0613}{12}\right)} \\ \therefore x &= \frac{750\ 000 \times \left(\frac{0{,}0613}{12}\right)}{\left[\left(1 + \frac{0{,}0613}{12}\right)^{(35\times 12)}\right]} \\ &= 510{,}84927\ldots \end{aligned}$$

In order to save R 750 000 in 35 years, Monique will need to save R 510,85 in her account every month.

b) How much money, to the nearest rand, did Monique deposit into her account by the end of the 35 year period?

Solution:

The final amount calculated in the question above includes the money Monique deposited into the account plus the interest paid by the bank. The total amount of money Monique put into her account during the 35 year is the product of 12 payments per year, 35 years, and the payment amount itself:

$$\text{R } 510{,}85 \times 12 \times 35 = \text{R } 214\ 557{,}00$$

After 35 years, Monique deposited a total of R 214 557 into her account.

6. Lerato plans to buy a car in five and a half years' time. She has saved R 30 000 in a separate investment account which earns 13% per annum compound interest. If she doesn't want to spend more than R 160 000 on a vehicle and her savings account earns an interest rate of 11% p.a. compounded monthly, how much must she deposit into her savings account each month?

Solution:

First calculate the accumulated amount for the R 30 000 in Lerato's investment account:

$$A = P(1+i)^n$$

$$\begin{aligned} P &= 30\ 000 \\ i &= 0{,}13 \\ n &= 5{,}5 \end{aligned}$$

$$\begin{aligned} A &= 30\ 000(1+0{,}13)^{5{,}5} \\ &= \text{R } 58\ 756{,}06 \end{aligned}$$

In five and a half years' time, Lerato needs to have saved R $160\ 000 -$ R $58\ 756{,}06 =$ R $101\ 243{,}94$.

$$x = \frac{F \times i}{[(1+i)^n - 1]}$$

$$\begin{aligned} F &= 101\ 243{,}94 \\ i &= \frac{0{,}11}{12} \\ n &= 5{,}5 \times 12 = 66 \end{aligned}$$

Substitute the known values and calculate x:

$$\begin{aligned} x &= \frac{101\ 243{,}94 \times \frac{0{,}11}{12}}{\left[(1+\frac{0{,}11}{12})^{66} - 1\right]} \\ &= \text{R } 1123{,}28 \end{aligned}$$

Write the final answer:

Lerato must deposit R 1123,28 each month into her savings account.

7. a) Every Monday Harold puts R 30 into a savings account at the King Bank, which accrues interest of 6,92% p.a. compounded weekly. How long will it take Harold's account to reach a balance of R 4397,53. Give the answer as a number of years and days to the nearest integer.

Solution:

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$
$$F = \text{R } 4397{,}53$$
$$x = \text{R } 30$$
$$i = 0{,}0692$$

$$4397{,}53 = \frac{(30)\left[\left(1+\frac{0{,}0692}{52}\right)^{(n\times 52)} - 1\right]}{\left(\frac{0{,}0692}{52}\right)}$$
$$4397{,}53 = \frac{(30)\left[(1{,}00133)^{52n} - 1\right]}{0{,}00133\ldots}$$

$$(0{,}00133\ldots)(4397{,}53) = (30)\left[(1{,}00133\ldots)^{52n} - 1\right]$$
$$\frac{5{,}85209\ldots}{30} = \left[(1{,}00133\ldots)^{52n} - 1\right]$$

$$0{,}19506\ldots + 1 = (1{,}00133\ldots)^{52n}$$
$$1{,}19506\ldots = (1{,}00133\ldots)^{52n}$$

Change to logarithmic form: $52n = \log_{1{,}00133\ldots}(1{,}19506\ldots)$
$$52n = 134$$
$$n = \frac{134}{52}$$
$$n = 2{,}57692\ldots$$

To get to the final answer for this question, convert $2{,}57692\ldots$ years into years and days.

$(0{,}57692\ldots) \times \frac{365}{\text{year}} = 210{,}577$ days

Harold's investment takes 2 years and 211 days to reach the final value of R $4397{,}53$.

b) How much interest will Harold receive from the bank during the period of his investment?

Solution:

The total amount Harold invests is as follows:

$30 \times 52 \times 2{,}57692\ldots = \text{R } 4020{,}00$

Therefore, the total amount of interest paid by the bank: R $4397{,}53$ − R $4020{,}00 =$ R $377{,}53$.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28JY 1b. 28JZ 1c. 28K2 2. 28K3 3. 28K4 4. 28K5
5. 28K6 6. 28K7 7. 28K8

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Exercise 3 – 3: Sinking funds

1. Mfethu owns his own delivery business and he will need to replace his truck in 6 years' time. Mfethu deposits R 3100 into a sinking fund each month, which earns 5,3% interest p.a. compounded monthly.

   a) How much money will be in the fund in 6 years' time, when Mfethu wants to buy the new truck?

   Solution:

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

Where:

$$x = 3100$$
$$i = 0{,}053$$
$$n = 6$$

Interest is compounded monthly: $i = 0{,}053 \rightarrow \frac{0{,}053}{12}$ and $n = 6 \rightarrow 6 \times (12)$.

$$F = \frac{(3100)\left[\left(1+\frac{0{,}053}{12}\right)^{(6\times 12)} - 1\right]}{\left(\frac{0{,}053}{12}\right)}$$
$$= \text{R } 262\ 094{,}55$$

After 6 years, Mfethu will have R 262 094,55 in his sinking fund.

   b) If a new truck costs R 285 000 in 6 years' time, will Mfethu have enough money to buy it?

   Solution:

   No, Mfethu does not have enough money in his account:

$$\text{R } 285\ 000 - \text{R } 262\ 094{,}55 = \text{R } 22\ 905{,}45$$

2. Atlantic Transport Company buys a van for R 265 000. The value of the van depreciates on a reducing-balance basis at 17% per annum. The company plans to replace this van in five years' time and they expect the price of a new van to increase annually by 12%.

   a) Calculate the book value of the van in five years' time.

   Solution:

$$P = 265\ 000$$
$$i = 0{,}17$$
$$n = 5$$

$$A = P(1-i)^n$$
$$= 265\ 000(1-0{,}17)^5$$
$$= \text{R } 104\ 384{,}58$$

   b) Determine the amount of money needed in the sinking fund for the company to be able to afford a new van in five years' time.

   Solution:

$$P = 265\ 000$$
$$i = 0{,}12$$
$$n = 5$$

$$A = P(1+i)^n$$
$$= 265\ 000(1+0{,}12)^5$$
$$= \text{R } 467\ 020{,}55$$

Therefore, the balance of the sinking fund (F) must be greater than the cost of a new van in five years' time less the money from the sale of the old van:

$$F = \text{R } 467\ 020{,}55 - \text{R } 104\ 384{,}58$$
$$= \text{R } 362\ 635{,}97$$

c) Calculate the required monthly deposits if the sinking fund earns an interest rate of 11% per annum compounded monthly.

Solution:

Calculate the required monthly payment into the sinking fund:

$$x = \frac{F \times i}{[(1+i)^n - 1]}$$

$$F = 362\ 635{,}97$$
$$i = \frac{0{,}11}{12}$$
$$n = 5 \times 12 = 60$$

Substitute the values and calculate x:

$$x = \frac{362\ 635{,}97 \times \frac{0{,}11}{12}}{\left[(1+\frac{0{,}11}{12})^{60} - 1\right]}$$
$$= \text{R } 4560{,}42$$

Therefore, the company must deposit R 4560,42 each month.

3. Tonya owns Freeman Travel Company and she will need to replace her computer in 7 years' time. Tonya creates a sinking fund so that she will be able to afford a new computer, which will cost R 8450. The sinking fund earns interest at a rate of 7,67% p.a. compounded each quarter.

a) How much money must Tonya save quarterly so that there will be enough money in the account to buy the new computer?

Solution:

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

Where:

$$F = 8450$$
$$i = 0{,}0767$$
$$n = 7$$

Interest is compounded per quarter, therefore $i = 0{,}0767 \rightarrow \frac{0{,}0767}{4}$ and $n = 7 \rightarrow 7 \times 4$

$$8450 = \frac{x\left[\left(1+\frac{0{,}0767}{4}\right)^{(7\times4)} - 1\right]}{\left(\frac{0{,}0767}{4}\right)}$$
$$\therefore x = \frac{\left(8450 \times \frac{0{,}0767}{4}\right)}{\left[\left(1+\frac{0{,}0767}{4}\right)^{(7\times4)} - 1\right]}$$
$$= 230{,}80273\ldots$$

Tonya must deposit R 230,80 into the sinking fund quarterly.

b) How much interest (to the nearest rand) does the bank pay into the account by the end of the 7 year period?

Solution:

Total savings:

$$\text{R } 230{,}80 \times 4 \times 7 = \text{R } 6462{,}40$$

Interest earned:

$$\text{R } 8450 - \text{R } 6462{,}40 = \text{R } 1987{,}60$$

To the nearest rand, the bank paid R 1988 into the account.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28K9 2. 28KB 3. 28KC

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3.4 Present value annuities

Exercise 3 – 4: Present value annuities

1. A property costs R 1 800 000. Calculate the monthly repayments if the interest rate is 14% p.a. compounded monthly and the loan must be paid off in 20 years' time.

   Solution:

   Write down the given information and the present value formula:

   $$P = \frac{x\left[1 - (1+i)^{-n}\right]}{i}$$

   To determine the monthly repayment amount, we make x the subject of the formula:

   $$x = \frac{P \times i}{[1 - (1+i)^{-n}]}$$

   $$P = \text{R } 1\ 800\ 000$$
   $$i = \frac{0{,}14}{12}$$
   $$n = 20 \times 12 = 240$$

   Substitute the known values and calculate x:

   $$x = \frac{1\ 800\ 000 \times \frac{0{,}14}{12}}{\left[1 - \left(1 + \frac{0{,}14}{12}\right)^{-240}\right]} = \text{R } 22\ 383{,}37$$

2. A loan of R 4200 is to be returned in two equal annual installments. If the rate of interest is 10% compounded annually, calculate the amount of each installment.

   Solution:

   Write down the given information and the present value formula:

   $$P = \frac{x\left[1 - (1+i)^{-n}\right]}{i}$$

To determine the monthly repayment amount, we make x the subject of the formula:

$$x = \frac{P \times i}{[1 - (1 + i)^{-n}]}$$

$$\begin{aligned} P &= \text{R } 4200 \\ i &= 0{,}10 \\ n &= 2 \end{aligned}$$

Substitute the known values and calculate x:

$$\begin{aligned} x &= \frac{4200 \times 0{,}10}{\left[1 - (1 + 0{,}10)^{-2}\right]} \\ &= \text{R } 2420{,}00 \end{aligned}$$

3. Stefan and Marna want to buy a flat that costs R 1,2 million. Their parents offer to put down a 20% payment towards the cost of the house. They need to get a mortgage for the balance. What is the monthly repayment amount if the term of the home loan is 30 years and the interest is 7,5% p.a. compounded monthly?

   Solution:

   Write down the given information and the present value formula:

$$P = \frac{x\left[1 - (1 + i)^{-n}\right]}{i}$$

   To determine the monthly repayment amount, we make x the subject of the formula:

$$x = \frac{P \times i}{[1 - (1 + i)^{-n}]}$$

$$\begin{aligned} P &= \text{R } 1\ 200\ 000 - \left(\text{R } 1\ 200\ 000 \times \frac{20}{100}\right) \\ &= \text{R } 960\ 000 \\ i &= \frac{0{,}075}{12} \\ n &= 30 \times 12 = 360 \end{aligned}$$

   Substitute the known values and calculate x:

$$\begin{aligned} x &= \frac{960\ 000 \times \frac{0{,}075}{12}}{\left[1 - \left(1 + \frac{0{,}075}{12}\right)^{-360}\right]} \\ &= \text{R } 6712{,}46 \end{aligned}$$

4. a) Ziyanda arranges a bond for R 17 000 from Langa Bank. If the bank charges 16,0% p.a. compounded monthly, determine Ziyanda's monthly repayment if she is to pay back the bond over 9 years.

   Solution:

$$P = \frac{x\left[1 - (1 + i)^{-n}\right]}{i}$$

   Where:

$$\begin{aligned} P &= 17\ 000 \\ i &= 0{,}16 \\ n &= 9 \end{aligned}$$

$$17\ 000 = \frac{x\left[1-\left(1+\frac{0,16}{12}\right)^{-(9\times 12)}\right]}{\left(\frac{0,16}{12}\right)}$$

$$\therefore x = \frac{17\ 000 \times \left(\frac{0,16}{12}\right)}{\left[1-\left(1+\frac{0,16}{12}\right)^{-(9\times 12)}\right]}$$

$$= 297,93$$

Ziyanda must pay R 297,93 each month.

b) What is the total cost of the bond?

Solution:

Total cost: R $297,93 \times 12 \times 9 =$ R 32 176,44

Ziyanda paid the bank a total of R 32 176,44.

5. Dullstroom Bank offers personal loans at an interest rate of 15,63% p.a. compounded twice a year. Lubabale borrows R 3000 and must pay R 334,93 every six months until the loan is fully repaid.

   a) How long will it take Lubabale to repay the loan?

   Solution:

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$P = 3000$$

$$x = 334,93$$

$$i = 0,1563$$

$$3000 = \frac{334,93\left[1-\left(1+\frac{0,1563}{2}\right)^{-(n\times 2)}\right]}{\left(\frac{0,1563}{2}\right)}$$

$$3000 = \frac{334,93\left[1-(1,07815\ldots)^{-2n}\right]}{0,07815\ldots}$$

$$(0,07815\ldots)(3000) = 334,93\left[1-(1,07815\ldots)^{-2n}\right]$$

$$\frac{(234,45)}{(334,93)} = 1-(1,07815\ldots)^{-2n}$$

$$0,69999\ldots = 1-(1,07815\ldots)^{-2n}$$

$$-0,30001\ldots = -(1,07815\ldots)^{-2n}$$

$$0,3\ldots = (1,07815\ldots)^{-2n}$$

$$-2n = \log_{1,07815\ldots}(0,3\ldots)$$

$$-2n = -16$$

$$n = \frac{-16}{-2}$$

$$n = 8$$

   It will take 8 years.

   b) How much interest will Lubabale pay for this loan?

   Solution:

   The total cost of the loan: $334,93 \times 2 \times 8 =$ R 5358,88.

   The total amount of interest Lubabale pays is: $5358,88 - 3000 =$ R 2358,88

6. Likengkeng has just started a new job and wants to buy a car that costs R 232 000. She visits the Soweto Savings Bank, where she can arrange a loan with an interest rate of 15,7% p.a. compounded monthly. Likengkeng has enough money saved to pay a deposit of R 50 000. She arranges a loan for the balance of the payment, which is to be paid over a period of 6 years.

   a) What is Likengkeng's monthly repayment on her loan?

   Solution:

   The balance of the payment: R 232 000 − R 50 000 = R 182 000

   Therefore, Likengkeng takes out a loan for R 182 000.

$$\begin{aligned} P &= \frac{x\left[1-(1+i)^{-n}\right]}{i} \\ P &= 182\ 000 \\ i &= 0{,}157 \\ n &= 6 \end{aligned}$$

$$\begin{aligned} 182\ 000 &= \frac{x\left[1-\left(1+\frac{0{,}157}{12}\right)^{-(6\times 12)}\right]}{\left(\frac{0{,}157}{12}\right)} \\ \therefore x &= \frac{182\ 000\times\left(\frac{0{,}157}{12}\right)}{\left[1-\left(1+\frac{0{,}157}{12}\right)^{-(6\times 12)}\right]} \\ &= \text{R } 3917{,}91 \end{aligned}$$

   Likengkeng must pay R 3917,91 each month.

   b) How much will the car cost Likengkeng?

   Solution:

   The total amount paid: R 3917,91 × 12 × 6 = R 282 089,87

   Therefore, Likengkeng paid a total of R 282 089,87 + R 50 000 = R 332 089,87

7. Anathi is a wheat farmer and she needs to buy a new holding tank which costs R 219 450. She bought her old tank 14 years ago for R 196 000. The value of the old grain tank has depreciated at a rate of 12,1% per year on a reducing balance, and she plans to trade it in for its current value. Anathi will then need to arrange a loan for the balance of the cost of the new grain tank.

   Orsmond bank offers loans with an interest rate of 9,71% p.a. compounded monthly for any loan up to R 170 000 and 9,31% p.a. compounded monthly for a loan above that amount. The loan agreement allows Anathi a grace period for the first six months (no payments are made) and it states that the loan must be repaid over 30 years.

   a) Determine the monthly repayment amount.

   Solution:

   Anathi will trade in the old grain tank to offset some of the cost of the new one. Therefore, we need to determine the current value of the old grain tank:

$$\begin{aligned} A &= P(1-i)^n \\ &= 196\ 000(1-0{,}121)^{14} \\ &= 196\ 000(0{,}16437\ldots) \\ &= 32\ 218{,}12946\ldots \end{aligned}$$

   The value of the old grain tank for the trade in is R 32 218,13.

   Therefore, the loan amount for the new tank is:

   R 219 450 − R 32 218,13 = R 187 231,87

   Determine which interest rate to use: the loan amount is more than R 170 000,00, therefore Anathi gets the lower interest rate of 9,31%.

   Calculate how much the loan will be worth at the end of the grace period:

$$\begin{aligned} A &= P(1+i)^n \\ &= 187\ 231{,}87\left(1+\frac{0{,}0931}{12}\right)^6 \\ &= 196\ 118{,}31962\ldots \end{aligned}$$

After the first six months of the loan period, the amount she owes increases to R 196 118,32.

Now we can use the present value formula to solve for the value of x. Remember that the time period for this calculation is 29,5 years.

$$\begin{aligned} 196\ 118{,}32 &= \frac{x\left[1-\left(1+\frac{0{,}0931}{12}\right)^{-(29{,}5\times 12)}\right]}{\left(\frac{0{,}0931}{12}\right)} \\ x &= \frac{196\ 118{,}32 \times \frac{0{,}0931}{12}}{\left[1-\left(1+\frac{0{,}0931}{12}\right)^{-(29{,}5\times 12)}\right]} \\ x &= 1627{,}0471\ldots \end{aligned}$$

Therefore, Anathi must pay R 1627,05 each month.

b) What is the total amount of interest Anathi will pay for the loan?

Solution:

The total cost of the loan is:

$$\text{R } 1627{,}05 \times 12 \times 29{,}5 = \text{R } 575\ 975{,}70$$

Therefore, the amount of interest is:

$$\text{R } 575\ 975{,}70 - \text{R } 187\ 231{,}87 = \text{R } 388\ 743{,}83$$

The total amount of interest Anathi paid is R 388 743,83.

c) How much money would Anathi have saved if she did not take the six month grace period?

Solution:

For this calculation the loan amount is R 187 231,87 and the time period is 30 years:

$$\begin{aligned} 187\ 231{,}87 &= \frac{x\left[1-\left(1+\frac{0{,}0931}{12}\right)^{-(30\times 12)}\right]}{\left(\frac{0{,}0931}{12}\right)} \\ x &= \frac{187\ 231{,}87 \times \frac{0{,}0931}{12}}{\left[1-\left(1+\frac{0{,}0931}{12}\right)^{-360}\right]} \\ x &= 1548{,}458\ldots \end{aligned}$$

Therefore, Anathi would pay R 1548,46 each month.

The total cost of the loan would be:

$$\text{R } 1548{,}46 \times 12 \times 30 = \text{R } 557\ 445{,}60$$

The amount of interest:

$$\text{R } 557\ 445{,}60 - \text{R } 187\ 231{,}87 = \text{R } 370\ 213{,}73$$

Anathi would have saved:

$$\text{R } 388\ 743{,}83 - \text{R } 370\ 213{,}73 = \text{R } 18\ 530{,}10$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28KD 2. 28KF 3. 28KG 4. 28KH 5. 28KJ 6. 28KK
7. 28KM

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Exercise 3 – 5: Analysing investment and loan options

1. Cokisa is 31 years old and starting to plan for her future. She has been thinking about her retirement and wants to open an annuity so that she will have money when she retires. Her intention is to retire when she is 65 years old. Cokisa visits the Trader's Bank of Tembisa and learns that there are two investment options from which she can choose:
   - Option A: 7,76% p.a. compounded once every four months
   - Option B: 7,78% p.a. compounded half-yearly

   a) Which is the better investment option for Cokisa if the amount she will deposit will always be the same?

   Solution:

   There are two separate account options for Cokisa to consider; to determine which is the better option we need to compare the effective interest of each option. The higher the effective interest, the quicker the account will grow. The effective interest formula is:

$$i+1=\left(1+\frac{i^m}{m}\right)^m$$

   Where:

$$\begin{aligned} i &= \text{the effective interest rate} \\ i^m &= \text{the nominal interest rate} \\ m &= \text{the number of compounding periods each year} \end{aligned}$$

   Work out the effective interest for option A:

$$\begin{aligned} i &= \left(1+\frac{0{,}0776}{3}\right)^3-1 \\ &= 0{,}07962\ldots \end{aligned}$$

   This calculation shows that option A an effective interest rate of about 7,9625%. Now work out the effective interest for option B:

$$\begin{aligned} i &= \left(1+\frac{0{,}0778}{2}\right)^2-1 \\ &= 0{,}07931\ldots \end{aligned}$$

   For option B, the effective interest rate will be approximately 7,9313%.
   By comparing the two calculations above, we see that the option A is the better one.
   NOTE: It may seem that we could answer this question by picking an amount for the regular payment and then using the future value formula for each of the two options to see which one produces more money. However, this will not work because the compounding periods are different. If we want to work it out this way, we MUST adjust the payment value we use according to the compounding period, for example if we choose a regular payment of R 100 once every four months for option A, then we will have to use R 150 semi-annually for option B (because R $100\times 3=300$ is the same total amount per year as R $150\times 2=300$).

   b) Cokisa opens an account and starts saving R 4000 every four months. How much money (to the nearest rand) will she have saved when she reaches her planned retirement?

   Solution:

$$\begin{aligned} F &= \frac{4000\left[\left(1+\frac{0{,}0776}{3}\right)^{(34\times 3)}-1\right]}{\left(\frac{0{,}0776}{3}\right)} \\ &= \text{R }1\ 937\ 512{,}76 \end{aligned}$$

   Cokisa will have R 1 937 512,76 for her retirement.

2. Phoebe wants to take out a home loan of R 1,6 million. She approaches three different banks for their loan options:

   - Bank A offers a repayment period of 30 years and an interest rate of 12% per annum compounded monthly.
   - Bank B offers a repayment period of 20 years and an interest rate of 14% per annum compounded monthly.
   - Bank C offers a repayment period of 30 years and an interest rate of 14% per annum compounded monthly.

   If Phoebe intends to start her monthly repayments immediately, calculate which of the three options would be best for her.

   Solution:

$$x = \frac{P \times i}{[1-(1+i)^{-n}]}$$

Bank A:

$$\begin{aligned} x &= \frac{1\ 600\ 000 \times \frac{0.12}{12}}{\left[1-(1+\frac{0.12}{12})^{-360}\right]} \\ &= \text{R } 16\ 457{,}80 \\ \text{Total amount } (T): &= 30 \times 12 \times \text{R } 16\ 457{,}80 \\ &= \text{R } 5\ 924\ 808 \\ \text{Total interest } (I): &= \text{R } 5\ 924\ 808 - \text{R } 1\ 600\ 000 \\ &= \text{R } 4\ 324\ 808 \end{aligned}$$

Bank B:

$$\begin{aligned} x &= \frac{1\ 600\ 000 \times \frac{0.14}{12}}{\left[1-(1+\frac{0.14}{12})^{-240}\right]} \\ &= \text{R } 19\ 896{,}33 \\ \text{Total amount } (T): &= 20 \times 12 \times \text{R } 19\ 896{,}33 \\ &= \text{R } 4\ 775\ 119{,}20 \\ \text{Total interest } (I): &= \text{R } 4\ 775\ 119{,}20 - \text{R } 1\ 600\ 000 \\ &= \text{R } 3\ 175\ 119{,}20 \end{aligned}$$

Bank C:

$$\begin{aligned} x &= \frac{1\ 600\ 000 \times \frac{0.14}{12}}{\left[1-(1+\frac{0.14}{12})^{-360}\right]} \\ &= \text{R } 18\ 957{,}95 \\ \text{Total amount } (T): &= 30 \times 12 \times \text{R } 18\ 957{,}95 \\ &= \text{R } 6\ 824\ 862 \\ \text{Total interest } (I): &= \text{R } 6\ 824\ 862 - \text{R } 1\ 600\ 000 \\ &= \text{R } 5\ 224\ 862 \end{aligned}$$

	x	T	I
Bank A	R 16 457,80	R 5 924 808,00	R 4 324 808,00
Bank B	R 19 896,33	R 4 775 119,20	R 3 175 119,20
Bank C	R 18 957,95	R 6 824 862,00	R 5 224 862,00

A loan from Bank A would have the lowest monthly repayments, however the interest paid is high as a result of the longer repayment period. Therefore, Phoebe should consider taking out a loan with Bank B, as this has the lowest total repayment amount.

Check answers online with the exercise code below or click on 'show me the answer'.
1. 28KN 2. 28KP

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3.6 Summary

Exercise 3 – 6: End of chapter exercises

1. Mpumelelo deposits R 500 into a savings account, which earns interest at 6,81% p.a. compounded quarterly. How long will it take for the savings account to have a balance of R 749,77?

   Solution:

$$A = P(1+i)^n$$

Where:

$$A = 749{,}77$$
$$P = 500$$
$$i = 0{,}0681$$

   In this question the interest is payable quarterly, so $i \rightarrow \frac{0{,}0681}{4}$ and $n \rightarrow (n \times 4)$. In this case, n represents the number of years; the product $(n \times 4)$ represents the number of times the bank pays interest into the account.

$$749{,}77 = 500\left(1 + \frac{0{,}0681}{4}\right)^{(n\times 4)}$$

$$\begin{aligned} 749{,}77 &= 500(1{,}01702\ldots)^{4n} \\ \frac{749{,}77}{500} &= (1{,}01702\ldots)^{4n} \\ 1{,}49954\ldots &= (1{,}01702\ldots)^{4n} \end{aligned}$$

   At this point we must change the equation to logarithmic form:

$$\begin{aligned} 4n &= \log_{1{,}017025}(1{,}49954\ldots) \\ 4n &= \frac{\log 1{,}49954\ldots}{\log 1{,}017025} \quad \text{(change of base)} \\ 4n &= 24 \end{aligned}$$

   To find the number of years, we solve for n:

$$\begin{aligned} 4n &= 24 \\ n &= \frac{24}{4} \\ n &= 6 \end{aligned}$$

   Therefore, it will take 6 years.

2. How much interest will Gavin pay on a loan of R 360 000 for 5 years at 10,3% per annum compounded monthly?

Solution:

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$\begin{aligned}
360\ 000 &= \frac{x\left[1-\left(1+\frac{0{,}103}{12}\right)^{-(5\times 12)}\right]}{\left(\frac{0{,}103}{12}\right)} \\
\therefore x &= \frac{360\ 000\times\left(\frac{0{,}103}{12}\right)}{\left[1-\left(1+\frac{0{,}103}{12}\right)^{-60}\right]} \\
&= 7702{,}184\ldots
\end{aligned}$$

Monthly payments are R 7702,18.

Total loan:

$$\text{R } 7702{,}18\times 12\times 5 = \text{R } 462\ 130{,}80$$

Interest:

$$\text{R } 462\ 130{,}80 - \text{R } 360\ 000 = \text{R } 102\ 130{,}80$$

3. Wingfield school will need to replace a number of old classroom desks in 6 years' time. The principal has calculated that the new desks will cost R 44 500. The school establishes a sinking fund to pay for the new desks and immediately deposits an amount of R 6300 into the fund, which accrues interest at a rate of 6,85% p.a. compounded monthly.

   a) How much money should the school save every month so that the sinking fund will have enough money to cover the cost of the desks?

   Solution:
   R 6300 is deposited into the sinking fund immediately and will earn interest until the end of the 6 years:

$$\begin{aligned}
A &= P(1+i)^n \\
&= 6300\left(1+\frac{0{,}0685}{12}\right)^{(6\times 12)} \\
&= 6300(1{,}0057\ldots)^{72} \\
&= 6300(1{,}50656\ldots) \\
&= 9491{,}35
\end{aligned}$$

   After 6 years the deposit will be worth R 9491,35.
   Balance required in the sinking fund:

$$\text{R } 44\ 500 - \text{R } 9491{,}35 = \text{R } 35\ 008{,}65$$

   We calculate the monthly payments that will give a future value of R 35 008,65:

$$\begin{aligned}
F &= \frac{x\left[(1+i)^n-1\right]}{i} \\
35\ 008{,}65 &= \frac{x\left[\left(1+\frac{0{,}0685}{12}\right)^{(6\times 12)}-1\right]}{\left(\frac{0{,}0685}{12}\right)} \\
\therefore x &= \frac{35\ 008{,}65\times\frac{0{,}0685}{12}}{\left[\left(1+\frac{0{,}0685}{12}\right)^{(6\times 12)}-1\right]} \\
&= 394{,}50326\ldots
\end{aligned}$$

   Therefore, Wingfield school must deposit R 394,50 into the sinking fund each month so that there will enough money in the account to buy the new desks.

   b) How much interest does the fund earn over the period of 6 years?

Solution:

$$\begin{aligned} \text{Total amount saved: } &= 6300 + (394{,}5 \times 12 \times 6) \\ &= 6300 + 28\ 404 \\ &= 34\ 704 \end{aligned}$$

The total amount of money the school deposits into the account is R 34 704,00. Therefore, the amount of interest paid by the bank:

$$\text{R } 44\ 500 - \text{R } 34\ 704{,}00 = \text{R } 9796{,}00$$

4. Determine how many years (to the nearest integer) it will take for the value of a motor vehicle to decrease to 25% of its original value if the rate of depreciation, based on the reducing-balance method, is 21% per annum.

Solution:

Let the value of the vehicle be x.

$$\begin{aligned} A &= P(1-i)^n \\ x \times \frac{25}{100} &= x\left(1 - \frac{21}{100}\right)^n \\ 0{,}25 &= \left(1 - \frac{21}{100}\right)^n \\ 0{,}25 &= (0{,}79)^n \\ \therefore n &= \log_{0{,}79} 0{,}25 \qquad \text{(use definition)} \\ &= \frac{\log 0{,}25}{\log 0{,}79} \qquad \text{(change of base)} \\ &= 5{,}881\ldots \end{aligned}$$

Therefore, it will take about 6 years.

5. Angela has just started a new job, and wants to save money for her retirement. She decides to deposit R 1300 into a savings account once each month. Her money goes into an account at Pinelands Mutual Bank, and the account receives 6,01% interest p.a. compounded once each month.

   a) How much money will Angela have in her account after 30 years?

   Solution:

$$\begin{aligned} F &= \frac{x\left[(1+i)^n - 1\right]}{i} \\ x &= \text{R } 1300 \\ i &= 0{,}0601 \\ n &= 30 \end{aligned}$$

$$\begin{aligned} F &= \frac{(1300)\left[\left(1 + \frac{0{,}0601}{12}\right)^{(30\times 12)} - 1\right]}{\left(\frac{0{,}0601}{12}\right)} \\ &= \text{R } 1\ 308\ 370{,}14 \end{aligned}$$

   After 30 years, Angela will have R 1 308 370,14 in her account.

   b) How much money did Angela deposit into her account after 30 years?

   Solution:

   The total amount of money Angela saves each year is $1300 \times 12 =$ R 15 600. From that we can determine the total amount she saves by multiplying by the number of years: $15\ 600 \times 30 =$ R 468 000.

   After 30 years, Angela deposited a total of R 468 000 into her account.

6. a) Nicky has been working at Meyer and Associates for 5 years and gets an increase in her salary. She opens a savings account at Langebaan Bank and begins making deposits of R 350 every month. The account earns 5,53% interest p.a., compounded monthly. Her plan is to continue saving on a monthly schedule until she retires. However, after 8 years she stops making the monthly payments and leaves the account to continue growing.

How much money will Nicky have in her account 29 years after she first opened it?

Solution:

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$
$$x = \text{R } 350$$
$$i = 0{,}0553$$
$$n = 8$$

$$F = \frac{(350)\left[\left(1 + \frac{0{,}0553}{12}\right)^{(8\times 12)} - 1\right]}{\left(\frac{0{,}0553}{12}\right)}$$
$$= \text{R } 42\ 141{,}06$$

This calculation shows that the account will have a balance of R 42 141,06 after 8 years, which is when Nicky stops making monthly deposits.

From this point on, the account grows from R 42 141,06 with compound interest only (no more monthly payments). This will continue for $29 - 8 = 21$ years.

$$A = P(1+i)^n$$
$$= 42\ 141{,}06\left(1 + \frac{0{,}0553}{12}\right)^{21\times 12}$$
$$= 134\ 243{,}45$$

The total amount of money in the account 29 years after Nicky opens the account is R 134 243,45.

b) Calculate the difference between the total deposits made into the account and the amount of interest paid by the bank.

Solution:

The total deposits:

$$\text{R } 350 \times 12 \times 8 = \text{R } 33\ 600$$

At the end of the period, the interest earned is:

$$\text{R } 134\ 243{,}45 - 33\ 600 = \text{R } 100\ 643{,}45$$

Difference:

$$\text{R } 100\ 643{,}45 - \text{R } 33\ 600 = \text{R } 67\ 043{,}45$$

7. a) Every three months Louis puts R 500 into an annuity. His account earns an interest rate of 7,51% p.a. compounded quarterly. How long will it take Louis's account to reach a balance of R 13 465,87?

Solution:

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$
$$F = \text{R } 13\ 465{,}87$$
$$x = \text{R } 500$$
$$i = 0{,}0751$$

$$13\ 465{,}87 = \frac{500\left[\left(1 + \frac{0{,}0751}{4}\right)^{(n\times 4)} - 1\right]}{\left(\frac{0{,}0751}{4}\right)}$$
$$13\ 465{,}87 = \frac{500\left[(1{,}01877\ldots)^{4n} - 1\right]}{0{,}01877\ldots}$$

$$(0{,}01877\ldots)(13\ 465{,}87) = (500)\left[(1{,}01877\ldots)^{4n} - 1\right]$$
$$\frac{252{,}82172\ldots}{500} = \left[(1{,}01877\ldots)^{4n} - 1\right]$$

Now that the square bracket is alone, work inside of it: add one to both sides, and then change the equation to log-form to get the exponent.

$$0{,}50564\ldots + 1 = (1{,}01877\ldots)^{4n}$$
$$1{,}50564\ldots = (1{,}01877\ldots)^{4n}$$

Change into logarithmic form: $4n = \log_{1{,}01877\ldots}(1{,}50564\ldots)$

$$4n = 22$$
$$n = \frac{22}{4}$$
$$n = 5{,}5$$

It will take about $5{,}5$ years.

b) How much interest will Louis receive from his investment?

Solution:

The total amount of deposits:

$$500 \times 4 \times 5{,}5 = \text{R } 11\ 000{,}00$$

The total amount of interest:

$$\text{R } 13\ 465{,}87 - \text{R } 11\ 000{,}00 = \text{R } 2465{,}87$$

8. A dairy farmer named Kayla needs to buy new equipment for her dairy farm which costs R $200\ 450$. She bought her old equipment 12 years ago for R $167\ 000$. The value of the old equipment depreciates at a rate of $12{,}2\%$ per year on a reducing balance. Kayla will need to arrange a bond for the remaining cost of the new equipment.

An agency which supports farmers offers bonds at a special interest rate of $10{,}01\%$ p.a. compounded monthly for any loan up to R $175\ 000$ and $9{,}61\%$ p.a. compounded monthly for a loan above that amount. Kayla arranges a bond such that she will not need to make any payments on the loan in the first six months (called a 'grace period') and she must pay the loan back over 20 years.

a) Determine the monthly payment.

Solution:

Old equipment:

$$\begin{aligned} A &= P(1 - i)^n \\ &= 167\ 000(1 - 0{,}122)^{12} \\ &= 35\ 046{,}98494\ldots \end{aligned}$$

The value of the old equipment is R $35\ 046{,}98$.

Bond amount:

$$\text{R } 200\ 450 - \text{R } 35\ 046{,}98 = \text{R } 165\ 403{,}02$$

$$\begin{aligned} A &= P(1 + i)^n \\ &= 165\ 403{,}02\left(1 + \frac{0{,}1001}{12}\right)^6 \\ &= 173\ 856{,}01291\ldots \end{aligned}$$

After the first six months of the loan period, during which she makes no payments, the amount she owes increases to R $173\ 856{,}01$.

Now we can use the present value formula to solve for the value of x. Remember that the repayment period is $19{,}5$ years.

$$173\ 856{,}01 = \frac{x\left[1-\left(1+\frac{0{,}1001}{12}\right)^{-(19{,}5\times 12)}\right]}{\left(\frac{0{,}1001}{12}\right)}$$
$$\therefore x = \frac{173\ 856{,}01 \times \left(\frac{0{,}1001}{12}\right)}{x\left[1-\left(1+\frac{0{,}1001}{12}\right)^{-(19{,}5\times 12)}\right]}$$
$$= 1692{,}53772\ldots$$

Kayla must pay R $1692{,}54$ each month.

b) What is the total amount of interest Kayla will pay for the bond?

Solution:

Total cost of the loan:

R $1692{,}54 \times 12 \times 19{,}5 =$ R $396\ 054{,}36$

Interest:

R $396\ 054{,}36 -$ R $165\ 403{,}02 =$ R $230\ 651{,}34$

The total amount of interest Kayla paid is R $230\ 651{,}34$.

c) By what factor is the interest she pays greater than the value of the loan? Give the answer correct to one decimal place.

Solution:

$$k = \frac{\text{interest paid}}{\text{amount of the loan}} = \frac{230\ 651{,}34}{165\ 403{,}02} = 1{,}39448$$

The interest is greater than the loan amount by a factor of $1{,}4$.

9. Thabo invests R 8500 in a special banking product which will pay 1% per annum for 1 month, then 2% per annum for the next 2 months, then 3% per annum for the next 3 months, 4% per annum for the next 4 months, and 0% for the rest of the year. If the bank charges him R 75 to open the account, how much can he expect to get back at the end of the year?

Solution:

Subtract account fee from the investment amount: R 8500 - R 75 = R 8425

$$A = P(1+i)^n$$

$$\text{At } T_0: \quad A = 8425$$
$$\text{At } T_1: \quad A = 8425\left(1+\frac{0{,}01}{12}\right)^1$$
$$\text{At } T_3: \quad A = 8425\left(1+\frac{0{,}01}{12}\right)^1\left(1+\frac{0{,}02}{12}\right)^2$$
$$\text{At } T_6: \quad A = 8425\left(1+\frac{0{,}01}{12}\right)^1\left(1+\frac{0{,}02}{12}\right)^2\left(1+\frac{0{,}03}{12}\right)^3$$
$$\text{At } T_{10}: \quad A = 8425\left(1+\frac{0{,}01}{12}\right)^1\left(1+\frac{0{,}02}{12}\right)^2\left(1+\frac{0{,}03}{12}\right)^3\left(1+\frac{0{,}04}{12}\right)^4$$
$$\therefore \text{Final amount} = \text{R } 8637{,}98$$

10. Thabani and Lungelo are both using Harper Bank for their savings. Lungelo makes a deposit of x at an interest rate of i for six years. Three years after Lungelo made his first deposit, Thabani makes a deposit of $3x$ at an interest rate of 8% per annum. If after 6 years their investments are equal, calculate the value of i (correct to three decimal places). If the sum of their investment is R 20 000, determine how much Thabani earned in 6 years.

Solution:

$$A = P(1+i)^n$$

$$x(1+i)^6 = 3x\left(1+\frac{8}{100}\right)^3$$

$$x(1+i)^6 - 3x(1{,}08)^3 = 0$$

$$x\left((1+i)^6 - 3(1{,}08)^3\right) = 0$$

$$\therefore x = 0 \text{ or } (1+i)^6 - 3(1{,}08)^3 = 0$$

$$\text{If } x = 0,\ i \text{ can be any value,}$$

$$\therefore \text{not a valid solution}$$

$$\text{If } (1+i)^6 - 3(1{,}08)^3 = 0$$

$$(1+i)^6 = 3(1{,}08)^3$$

$$1+i = \sqrt[6]{3(1{,}08)^3}$$

$$\therefore i = \sqrt[6]{3(1{,}08)^3} - 1$$

$$= 0{,}248\ldots$$

$$\therefore i = 24{,}8\%$$

$$x + 3x = \text{R } 20\ 000$$

$$4x = \text{R } 20\ 000$$

$$\therefore x = \text{R } 5000$$

$$A = 15\ 000\left(1+\frac{8}{100}\right)^3$$

$$= 15\ 000(1{,}08)^3$$

$$= 18\ 895{,}68$$

$$\text{Interest earned:} = \text{R } 18\ 895{,}68 - \text{R } 15\ 000$$

$$= \text{R } 3895{,}68$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28KQ 2. 28KR 3. 28KS 4. 28KT 5. 28KV 6. 28KW
7. 28KX 8. 28KY 9. 28KZ 10. 28M2

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CHAPTER 4

Trigonometry

- Emphasize the value and importance of making sketches, where appropriate.
- It is very important for learners to understand that it is incorrect to apply the distributive law to the trigonometric ratios of compound angles and that $\cos(\alpha - \beta) \neq \cos\alpha - \cos\beta$.
- Emphasize that the area, sine and cosine rules do not require right-angled triangles.
- Remind learners that angles in the Cartesian plane are always measured from the positive x-axis.
- Important to note that $(270^\circ \pm x)$ is not included in the curriculum.
- Note that the co-function for tangent is also not included in the curriculum.
- Remind learners to check that their answers are within the required interval.
- For the general solution, determine the solution in the correct quadrants and within the required interval.
- To prove that an identity is true, remind learners that they are only allowed to manipulate **one side at a time**.
- To prove identities, we usually manipulate the more complicated expression so that it looks the same as the more simple expression.

4.1 Revision

Exercise 4 – 1: Revision - reduction formulae, co-functions and identities

1. Given: $\sin 31^\circ = A$
   Write each of the following expressions in terms of A:

   a) $\sin 149^\circ$
   Solution:

$$\begin{aligned} \sin 149^\circ &= \sin(180^\circ - 31^\circ) \\ &= \sin 31^\circ \\ &= A \end{aligned}$$

   b) $\cos(-59^\circ)$
   Solution:

$$\begin{aligned} \cos(-59^\circ) &= \cos 59^\circ \\ &= \cos(90^\circ - 31^\circ) \\ &= \sin 31^\circ \\ &= A \end{aligned}$$

   c) $\cos 329^\circ$
   Solution:

$$\begin{aligned} \cos 329^\circ &= \cos(360^\circ - 31^\circ) \\ &= \cos 31^\circ \\ &= \sqrt{1 - \sin^2 31^\circ} \\ &= \sqrt{1 - A^2} \end{aligned}$$

   d) $\tan 211^\circ \cos 211^\circ$

Solution:

$$\begin{aligned}\tan 211^\circ \cos 211^\circ &= \left(\frac{\sin 211^\circ}{\cos 211^\circ}\right)\cos 211^\circ \\ &= \sin 211^\circ \\ &= \sin(180^\circ + 31^\circ) \\ &= -\sin 31^\circ \\ &= -A\end{aligned}$$

e) $\tan 31^\circ$

Solution:

$$\begin{aligned}\tan 31^\circ &= \frac{\sin 31^\circ}{\cos 31^\circ} \\ &= \frac{A}{\sqrt{1-A^2}}\end{aligned}$$

2. a) Simplify P to a single trigonometric ratio:

$$P = \sin(360^\circ + \theta)\cos(180^\circ + \theta)\tan(360^\circ + \theta)$$

Solution:

$$\begin{aligned}P &= \sin(360^\circ + \theta)\cos(180^\circ + \theta)\tan(360^\circ + \theta) \\ &= \sin\theta(-\cos\theta)(\tan\theta) \\ &= \sin\theta\cos\theta\left(\frac{\sin\theta}{\cos\theta}\right) \\ &= \sin^2\theta\end{aligned}$$

b) Simplify Q to a single trigonometric ratio:

$$Q = \frac{\cos(\theta - 360^\circ)\sin(90^\circ + \theta)\sin(-\theta)}{\sin(\theta + 180^\circ)}$$

Solution:

Note: $\cos(\theta - 360^\circ)$

$$\begin{aligned}&= \cos[-(360^\circ - \theta)] \\ &= \cos(360^\circ - \theta) \\ &= \cos\theta\end{aligned}$$

$$\begin{aligned}Q &= \frac{\cos(\theta - 360^\circ)\sin(90^\circ + \theta)\sin(-\theta)}{\sin(\theta + 180^\circ)} \\ &= \frac{\cos\theta\cos\theta(-\sin\theta)}{-\sin\theta} \\ &= \cos^2\theta\end{aligned}$$

c) Hence, determine:

i. $P + Q$

ii. $\frac{Q}{P}$

Solution:

i.

$$\begin{aligned}P + Q &= \sin^2\theta + \cos^2\theta \\ &= 1\end{aligned}$$

ii.

$$\frac{Q}{P} = \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\tan^2\theta}$$

3. If $p = \sin\beta$, express the following in terms of p:

$$\frac{\cos(\beta + 360°)\tan(\beta - 360°)\cos(\beta + 90°)}{\sin^2(\beta + 180°)\cos(\beta - 90°)}$$

Solution:

Note: $\tan(\beta - 360°)$
$= \tan[-(360° - \beta)]$
$= -\tan(360° - \beta)$
$= -(-\tan\beta)$
$= \tan\beta$

And $\cos(\beta - 90°)$
$= \cos[-(90° - \beta)]$
$= \cos(90° - \beta)$
$= \sin\beta$

$$\begin{aligned}&\frac{\cos(\beta + 360°)\tan(\beta - 360°)\cos(\beta + 90°)}{\sin^2(\beta + 180°)\cos(\beta - 90°)}\\ &= \frac{\cos\beta\tan\beta(-\sin\beta)}{(-\sin\beta)^2\sin\beta}\\ &= -\frac{\cos\beta\left(\frac{\sin\beta}{\cos\beta}\right)\sin\beta}{(\sin^2\beta)\sin\beta}\\ &= -\frac{\sin^2\beta}{\sin^2\beta\sin\beta}\\ &= -\frac{1}{\sin\beta}\\ &= -\frac{1}{p}\end{aligned}$$

4. Evaluate the following without the use of a calculator:

a) $\dfrac{\cos(-120°)}{\tan 150°} + \cos 390°$

Solution:

$$\begin{aligned}&\frac{\cos(120°)}{\tan 150°} + \cos 390°\\ &= \frac{\cos(180° - 60°)}{\tan(180° - 30°)} + \cos(360° + 30°)\\ &= \frac{-\cos 60°}{-\tan 30°} + \cos 30°\\ &= \frac{\sin 30°}{\frac{\sin 30°}{\cos 30°}} + \cos 30°\\ &= \cos 30° + \cos 30°\\ &= 2\cos 30°\\ &= 2\left(\frac{\sqrt{3}}{2}\right)\\ &= \sqrt{3}\end{aligned}$$

b) $(1-\sin 45^\circ)(1-\sin 225^\circ)$

Solution:

$$\begin{aligned}
&(1-\sin 45^\circ)(1-\sin 225^\circ)\\
&=1-\sin 45^\circ-\sin 225^\circ+(\sin 45^\circ)(\sin 225^\circ)\\
&=1-\sin 45^\circ-\sin(180^\circ+45^\circ)+(\sin 45^\circ)(\sin(180^\circ+45^\circ))\\
&=1-\sin 45^\circ+\sin 45^\circ-\sin^2 45^\circ\\
&=1-\sin^2 45^\circ\\
&=1-\left(\frac{1}{\sqrt{2}}\right)^2\\
&=1-\frac{1}{2}\\
&=\frac{1}{2}
\end{aligned}$$

5. Reduce the following to one trigonometric ratio:

a) $\tan^2\beta-\dfrac{1}{\cos^2\beta}$

Solution:

$$\begin{aligned}
\tan^2\beta-\frac{1}{\cos^2\beta}&=\frac{\sin^2\beta}{\cos^2\beta}-\frac{1}{\cos^2\beta}\\
&=\frac{\sin^2\beta-1}{\cos^2\beta}\\
&=\frac{-(1-\sin^2\beta)}{\cos^2\beta}\\
&=\frac{-\cos^2\beta}{\cos^2\beta}\\
&=-1
\end{aligned}$$

b) $\sin^2(90^\circ+\theta)\tan^2\theta+\tan^2\theta\cos^2(90^\circ-\theta)$

Solution:

$$\begin{aligned}
&\sin^2(90^\circ+\theta)\tan^2\theta+\tan^2\theta\cos^2(90^\circ-\theta)\\
&=\cos^2\theta\tan^2\theta+\tan^2\theta\sin^2\theta\\
&=\tan^2\theta\left(\cos^2\theta+\sin^2\theta\right)\\
&=\tan^2\theta\,(1)\\
&=\tan^2\theta
\end{aligned}$$

c) $\sin\alpha\cos\alpha\tan\alpha-1$

Solution:

$$\begin{aligned}
\sin\alpha\cos\alpha\tan\alpha-1&=\sin\alpha\cos\alpha\left(\frac{\sin\alpha}{\cos\alpha}\right)-1\\
&=\sin^2\alpha-1\\
&=-\left(1-\sin^2\alpha\right)\\
&=-\cos^2\alpha
\end{aligned}$$

d) $\tan^2\theta+\dfrac{\cos^2\theta-1}{\cos^2\theta}$

Solution:

$$\begin{aligned}\tan^2\theta + \frac{\cos^2\theta - 1}{\cos^2\theta} &= \tan^2\theta - \frac{(1-\cos^2\theta)}{\cos^2\theta}\\ &= \tan^2\theta - \frac{\sin^2\theta}{\cos^2\theta}\\ &= \tan^2\theta - \tan^2\theta\\ &= 0\end{aligned}$$

6. a) Use reduction formulae and special angles to show that

$$\frac{\sin(180° + \theta)\tan(720° + \theta)\cos(-\theta)}{\cos(90° + \theta)}$$

can be simplified to $\sin\theta$.

Solution:

Use reduction formulae and co-functions to simplify the expression

$$\begin{aligned}&\frac{\sin(180° + \theta)\tan(720° + \theta)\cos(-\theta)}{\cos(90° + \theta)}\\ &= \frac{-\sin\theta\tan\left(2(360°) + \theta\right)\cos\theta}{-\sin\theta}\\ &= \tan\theta\cos\theta\\ &= \left(\frac{\sin\theta}{\cos\theta}\right)\cos\theta\\ &= \sin\theta\end{aligned}$$

b) Without using a calculator, determine the value of $\sin 570°$.

Solution:

Use special angles to determine the value of the expression

$$\begin{aligned}\sin 570° &= \sin(360° + 210°)\\ &= \sin(210°)\\ &= \sin(180° + 30°)\\ &= -\sin 30°\\ &= -\frac{1}{2}\end{aligned}$$

7. Troy's mathematics teacher asks the class to answer the following question.

Question:

Prove that $\frac{\cos\theta}{1+\sin\theta} = \frac{1-\sin\theta}{\cos\theta}$.

Troy's answer:

$$\begin{aligned}\frac{\cos\theta}{1+\sin\theta} &= \frac{1-\sin\theta}{\cos\theta}\\ (\cos\theta)(\cos\theta) &= (1+\sin\theta)(1-\sin\theta)\\ \cos^2\theta &= 1-\sin^2\theta\\ \cos^2\theta &= \cos^2\theta\\ \therefore \text{LHS} &= \text{RHS}\end{aligned}$$

Comment on Troy's answer and show the correct method for proving this identity.

Solution:

The question requires that Troy prove the identity. However, by working with both sides of the identity at the same time, he accepted that it was true. The correct method for proving an identity is to work with only one side at a time and to show that one side equals the other. Sometimes it is necessary to first simplify one side of the identity, and then also to simplify the other side in order to show that they are equal. Troy also should have stated restrictions.

Correct method:

$$\begin{aligned}\text{RHS} &= \frac{1-\sin\theta}{\cos\theta}\\ &= \frac{1-\sin\theta}{\cos\theta}\times\frac{1+\sin\theta}{1+\sin\theta}\\ &= \frac{1-\sin^2\theta}{\cos\theta(1+\sin\theta)}\\ &= \frac{\cos^2\theta}{\cos\theta(1+\sin\theta)}\\ &= \frac{\cos\theta}{1+\sin\theta}\\ &= \text{LHS}\end{aligned}$$

Restrictions: undefined where $\cos\theta = 0$, and $\sin\theta = -1$.

So then $\theta \neq 90 + k\ .\ 180°$ and $\theta \neq -90 + k\ .\ 360°$.

Therefore $\theta \neq 90° + k\ .\ 180°, \quad k \in \mathbb{Z}$.

8. Prove the following identities:

(State any restricted values in the interval $[0°; 360°]$, where appropriate.)

a) $\sin^2\alpha + (\cos\alpha - \tan\alpha)(\cos\alpha + \tan\alpha) = 1 - \tan^2\alpha$

Solution:

$$\begin{aligned}\text{LHS} &= \sin^2\alpha + (\cos\alpha - \tan\alpha)(\cos\alpha + \tan\alpha)\\ &= \sin^2\alpha + \cos^2\alpha - \tan^2\alpha\\ &= 1 - \tan^2\alpha\\ &= \text{RHS}\end{aligned}$$

Restrictions: undefined where $\tan\alpha$ is undefined.

Therefore $\alpha \neq 90°; 270°$.

b) $\frac{1}{\cos\theta} - \frac{\cos\theta\tan^2\theta}{1} = \cos\theta$

Solution:

$$\begin{aligned}\text{LHS} &= \frac{1}{\cos\theta} - \frac{\cos\theta\tan^2\theta}{1}\\ &= \frac{1-\cos^2\theta\times\tan^2\theta}{\cos\theta}\\ &= \frac{1-\cos^2\theta\times\frac{\sin^2\theta}{\cos^2\theta}}{\cos\theta}\\ &= \frac{1-\sin^2\theta}{\cos\theta}\\ &= \frac{\cos^2\theta}{\cos\theta}\\ &= \cos\theta\\ &= \text{RHS}\end{aligned}$$

Restrictions: undefined where $\cos\theta = 0$ and where $\tan\theta$ is undefined.

Therefore $\theta \neq 90°; 270°$.

c) $\frac{2\sin\theta\cos\theta}{\sin\theta+\cos\theta} = \sin\theta + \cos\theta - \frac{1}{\sin\theta+\cos\theta}$

Solution:

$$\begin{aligned}
\text{RHS} &= \sin\theta + \cos\theta - \frac{1}{\sin\theta + \cos\theta} \\
&= \frac{\sin^2\theta + \sin\theta\cos\theta + \cos\theta\sin\theta + \cos^2\theta - 1}{\sin\theta + \cos\theta} \\
&= \frac{1 + 2\sin\theta\cos\theta - 1}{\sin\theta + \cos\theta} \\
&= \frac{2\sin\theta\cos\theta}{\sin\theta + \cos\theta} \\
&= \text{LHS}
\end{aligned}$$

Restrictions: undefined where $\sin\theta = 0,\ \ \cos\theta = 0$.
Therefore $\theta \neq 0°; 90°; 180°; 270°; 360°$.

d) $\left(\frac{\cos\beta}{\sin\beta} + \tan\beta\right)\cos\beta = \frac{1}{\sin\beta}$

Solution:

$$\begin{aligned}
\text{LHS} &= \left(\frac{\cos\beta}{\sin\beta} + \frac{\sin\beta}{\cos\beta}\right)\cos\beta \\
&= \left(\frac{\cos^2\beta + \sin^2\beta}{\sin\beta\cos\beta}\right)\cos\beta \\
&= \frac{1}{\sin\beta} \\
&= \text{RHS}
\end{aligned}$$

Restrictions: undefined where $\sin\beta = 0,\ \ \cos\beta = 0$ and where $\tan\beta$ is undefined.
Therefore $\beta \neq 0°; 90°; 180°; 270°; 360°$.

e) $\frac{1}{1 + \sin\theta} + \frac{1}{1 - \sin\theta} = \frac{2\tan\theta}{\sin\theta\cos\theta}$

Solution:

$$\begin{aligned}
\text{LHS} &= \frac{1 - \sin\theta + 1 + \sin\theta}{(1 + \sin\theta)(1 - \sin\theta)} \\
&= \frac{2}{1 - \sin^2\theta} \\
&= \frac{2}{\cos^2\theta}
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \frac{2\tan\theta}{\sin\theta\cos\theta} \\
&= \frac{2\sin\theta}{\sin\theta\cos\theta\cos\theta} \\
&= \frac{2}{\cos^2\theta} \\
\therefore \text{LHS} &= \text{RHS}
\end{aligned}$$

Restrictions: undefined where $\sin\theta = \pm 1,\ \ \sin\theta = 0,\ \ \cos\theta = 0$.
Restrictions also include the values of θ for which $\tan\theta$ is undefined.
Therefore $\theta \neq 0°; 90°; 180°; 270°; 360°$.

f) $(1 + \tan^2\alpha)\cos\alpha = \frac{1 - \tan\alpha}{\cos\alpha - \sin\alpha}$

Solution:

$$\begin{aligned}
\text{LHS} &= (1+\tan^2\alpha)\cos\alpha \\
&= \left(1+\frac{\sin^2\alpha}{\cos^2\alpha}\right)\cos\alpha \\
&= \left(\frac{\cos^2\alpha+\sin^2\alpha}{\cos^2\alpha}\right)\cos\alpha \\
&= \frac{1}{\cos^2\alpha}\times\cos\alpha \\
&= \frac{1}{\cos\alpha}
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \frac{1-\tan\alpha}{\cos\alpha-\sin\alpha} \\
&= \frac{1-\frac{\sin\alpha}{\cos\alpha}}{\cos\alpha-\sin\alpha} \\
&= \frac{\frac{\cos\alpha-\sin\alpha}{\cos\alpha}}{\cos\alpha-\sin\alpha} \\
&= \frac{\cos\alpha-\sin\alpha}{\cos\alpha(\cos\alpha-\sin\alpha)} \\
&= \frac{1}{\cos\alpha} \\
&= \text{LHS}
\end{aligned}$$

Restrictions: where $\sin\alpha = \cos\alpha$ and where $\tan\alpha$ is undefined.
Therefore $\alpha \neq 45^\circ; 90^\circ; 270^\circ; 225^\circ$.

9. Determine whether the following statements are true or false.
If the statement is false, choose a suitable value between 0° and 90° to confirm your answer.

a) $\cos(180^\circ - \theta) = -1 - \cos\theta$
Solution:
False

$$\begin{aligned}
\text{Let } \theta &= 35^\circ \\
\text{LHS} &= \cos(180^\circ - 35^\circ) \\
&= \cos 145^\circ \\
&= -0{,}819
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= -1 - \cos 35^\circ \\
&= -1{,}819
\end{aligned}$$

$$\text{LHS} \neq \text{RHS}$$

b) $\sin(\alpha+\beta) = \sin\alpha + \sin\beta$
Solution:
False

$$\begin{aligned}
\text{Let } \alpha &= 62^\circ \\
\text{Let } \beta &= 20^\circ \\
\text{LHS} &= \sin(62^\circ + 20^\circ) \\
&= \sin 82^\circ \\
&= 0{,}990
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \sin 62^\circ + \sin 20^\circ \\
&= 1{,}224
\end{aligned}$$

$$\text{LHS} \neq \text{RHS}$$

c) $\sin \alpha = 2 \sin \frac{\alpha}{2} \sin \frac{\alpha}{2}$

Solution:

True

d) $\frac{1}{3} \sin 3\alpha = \sin \alpha$

Solution:

False

$$\text{Let } \alpha = 62^\circ$$
$$\text{LHS} = \frac{1}{3} \sin 3(62^\circ)$$
$$= -0{,}034$$
$$\text{RHS} = \sin 62^\circ$$
$$= 0{,}882$$
$$\text{LHS} \neq \text{RHS}$$

e) $\cos \beta = \sqrt{1 - \sin^2 \beta}$

Solution:

True

f) $\sin \theta = \tan \theta \cos \theta$

Solution:

True

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28M3	1b. 28M4	1c. 28M5	1d. 28M6	1e. 28M7	2. 28M8
3. 28M9	4a. 28MB	4b. 28MC	5a. 28MD	5b. 28MF	5c. 28MG
5d. 28MH	6. 28MJ	7. 28MK	8a. 28MM	8b. 28MN	8c. 28MP
8d. 28MQ	8e. 28MR	8f. 28MS	9a. 28MT	9b. 28MV	9c. 28MW
9d. 28MX	9e. 28MY	9f. 28MZ			

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4.2 Compound angle identities

Derivation of $\cos(\alpha - \beta)$

Exercise 4 – 2: Compound angle formulae

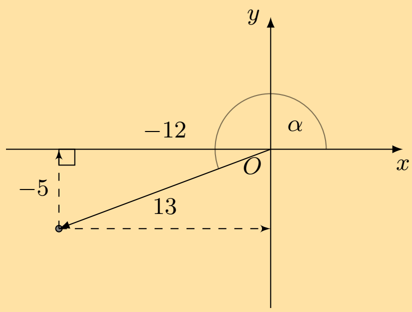
1. Given:

$$13 \sin \alpha + 5 = 0 \qquad (0^\circ < \alpha < 270^\circ)$$
$$13 \cos \beta - 12 = 0 \qquad (90^\circ < \beta < 360^\circ)$$

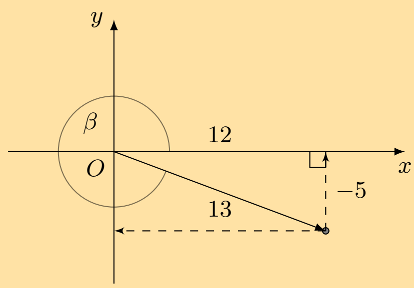
Draw a sketch and determine the following, without the use of a calculator:

a) $\tan \alpha - \tan \beta$

Solution:



$$
\begin{aligned}
13\sin\alpha + 5 &= 0 && (0^\circ < \alpha < 270^\circ)\\
\therefore \sin\alpha &= -\tfrac{5}{13} && \text{(third quadrant)}\\
\therefore x^2 &= (13)^2 - (-5)^2 && \text{(Pythagoras)}\\
&= 144\\
x &= \pm 12\\
\therefore x &= -12 && \text{(third quadrant)}
\end{aligned}
$$



$$
\begin{aligned}
13\cos\beta - 12 &= 0 && (90^\circ < \beta < 360^\circ)\\
\therefore \cos\beta &= \tfrac{12}{13} && \text{(fourth quadrant)}\\
\therefore y^2 &= (13)^2 - (12)^2 && \text{(Pythagoras)}\\
&= 25\\
y &= \pm 5\\
\therefore y &= -5 && \text{(fourth quadrant)}
\end{aligned}
$$

$$
\begin{aligned}
\tan\alpha - \tan\beta &= \frac{-5}{-12} - \left(\frac{-5}{12}\right)\\
&= \frac{5}{12} + \frac{5}{12}\\
&= \frac{10}{12}\\
&= \frac{5}{6}
\end{aligned}
$$

b) $\sin(\beta - \alpha)$

Solution:

$$
\begin{aligned}
\sin(\beta-\alpha) &= \sin\beta\cos\alpha - \cos\beta\sin\alpha\\
&= \frac{-5}{13}\cdot\frac{-12}{13} - \frac{12}{13}\cdot\frac{-5}{13}\\
&= \frac{60}{169} + \frac{60}{169}\\
&= \frac{120}{169}
\end{aligned}
$$

c) $\cos(\alpha + \beta)$

Solution:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ &= \frac{-12}{13}\cdot\frac{12}{13} - \frac{-5}{13}\cdot\frac{-5}{13}\\ &= -\frac{144}{169} - \frac{25}{169}\\ &= -\frac{169}{169}\\ &= -1\end{aligned}$$

2. Calculate the following without the use of a calculator (leave answers in surd form):

   a) $\sin 105^\circ$

      Solution:

$$\begin{aligned}\sin 105^\circ &= \sin(60^\circ + 45^\circ)\\ &= \sin 60^\circ\cos 45^\circ + \cos 60^\circ \sin 45^\circ\\ &= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}\end{aligned}$$

   b) $\cos 15^\circ$

      Solution:

$$\begin{aligned}\cos 15^\circ &= \cos(60^\circ - 45^\circ)\\ &= \cos 60^\circ\cos 45^\circ + \sin 60^\circ \sin 45^\circ\\ &= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{2}\left(1+\sqrt{3}\right)}{4}\end{aligned}$$

   c) $\sin 15^\circ$

      Solution:

$$\begin{aligned}\sin 15^\circ &= \sin(60^\circ - 45^\circ)\\ &= \sin 60^\circ\cos 45^\circ - \cos 60^\circ \sin 45^\circ\\ &= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}\end{aligned}$$

   d) $\tan 15^\circ$

Solution:

$$\begin{aligned}
\tan 15^\circ &= \frac{\sin 15^\circ}{\cos 15^\circ} \\
&= \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4} \div \frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4} \\
&= \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4} \times \frac{4}{\sqrt{2}\left(\sqrt{3}+1\right)} \\
&= \frac{\sqrt{3}-1}{\sqrt{3}+1} \\
&= \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} \\
&= \frac{3-2\sqrt{3}+1}{2} \\
&= \frac{4-2\sqrt{3}}{2} \\
&= 2-\sqrt{3}
\end{aligned}$$

e) $\cos 20^\circ \cos 40^\circ - \sin 20^\circ \sin 40^\circ$

Solution:

$$\begin{aligned}
&\cos 20^\circ \cos 40^\circ - \sin 20^\circ \sin 40^\circ \\
&= \cos(20^\circ + 40^\circ) \\
&= \cos 60^\circ \\
&= \frac{1}{2}
\end{aligned}$$

f) $\sin 10^\circ \cos 80^\circ + \cos 10^\circ \sin 80^\circ$

Solution:

$$\begin{aligned}
&\sin 10^\circ \cos 80^\circ + \cos 10^\circ \sin 80^\circ \\
&= \sin(10^\circ + 80^\circ) \\
&= \sin 90^\circ \\
&= 1
\end{aligned}$$

g) $\cos(45^\circ - x) \cos x - \sin(45^\circ - x) \sin x$

Solution:

$$\begin{aligned}
&\cos(45^\circ - x) \cos x - \sin(45^\circ - x) \sin x \\
&= \cos((45^\circ - x) + x) \\
&= \cos 45^\circ \\
&= \frac{1}{\sqrt{2}} \\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

h) $\cos^2 15^\circ - \sin^2 15^\circ$

Solution:

$$\begin{aligned}
&\cos^2 15^\circ - \sin^2 15^\circ \\
&= \cos 15^\circ \cos 15^\circ - \sin 15^\circ \sin 15^\circ \\
&= \cos(15^\circ + 15^\circ) \\
&= \cos 30^\circ \\
&= \frac{\sqrt{3}}{2}
\end{aligned}$$

3. a) Prove: $\sin(60° - x) + \sin(60° + x) = \sqrt{3}\cos x$

Solution:

$$\begin{aligned}
\text{LHS} &= \sin(60° - x) + \sin(60° + x) \\
&= \sin 60° \cos x - \cos 60° \sin x + \sin 60° \cos x + \cos 60° \sin x \\
&= 2 \sin 60° \cos x \\
&= 2\left(\frac{\sqrt{3}}{2}\right)\cos x \\
&= \sqrt{3}\cos x \\
&= \text{RHS}
\end{aligned}$$

b) Hence, evaluate $\sin 15° + \sin 105°$ without using a calculator.

Solution:

We have shown that:

$$\begin{aligned}
\sin(60° - x) + \sin(60° + x) &= \sqrt{3}\cos x \\
\text{If we let } x &= 45° \\
\sin(60° - 45°) + \sin(60° + 45°) &= \sqrt{3}\cos 45° \\
\sin 15° + \sin 105° &= \sqrt{3}\,.\,\frac{1}{\sqrt{2}} \\
&= \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{\sqrt{6}}{2}
\end{aligned}$$

c) Use a calculator to check your answer.

Solution:

For $x = 45°$:

$$\begin{aligned}
\text{LHS} &= \sin 15° + \sin 105° \\
&= 1{,}2247\ldots \\
\text{RHS} &= 1{,}2247\ldots \\
\therefore \text{LHS} &= \text{RHS}
\end{aligned}$$

4. Simplify the following without using a calculator:

$$\frac{\sin p \cos(45° - p) + \cos p \sin(45° - p)}{\cos p \cos(60° - p) - \sin p \sin(60° - p)}$$

Solution:

$$\begin{aligned}
&\frac{\sin p \cos(45° - p) + \cos p \sin(45° - p)}{\cos p \cos(60° - p) - \sin p \sin(60° - p)} \\
&= \frac{\sin[p + (45° - p)]}{\cos[p + (60° - p)]} \\
&= \frac{\sin 45°}{\cos 60°} \\
&= \frac{1}{\sqrt{2}} \div \frac{1}{2} \\
&= \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\
&= \sqrt{2}
\end{aligned}$$

5. a) Prove: $\sin(A + B) - \sin(A - B) = 2\cos A \sin B$

Solution:

$$\begin{aligned}
\text{LHS} &= \sin(A+B) - \sin(A-B) \\
&= \sin A\cos B + \cos A\sin B - [\sin A\cos B - \cos A\sin B] \\
&= \sin A\cos B + \cos A\sin B - \sin A\cos B + \cos A\sin B \\
&= 2\cos A\sin B \\
&= \text{RHS}
\end{aligned}$$

b) Hence, calculate the value of $\cos 75^\circ \sin 15^\circ$ without using a calculator.

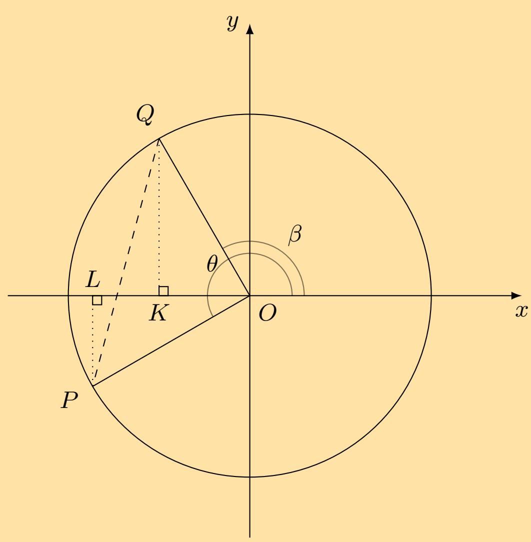
Solution:

We have shown that:

$$\begin{aligned}
2\cos A\sin B &= \sin(A+B) - \sin(A-B) \\
\therefore \cos A\sin B &= \frac{1}{2}\left(\sin(A+B) - \sin(A-B)\right) \\
\text{Let } A &= 75^\circ \\
\text{And let } B &= 15^\circ \\
\cos 75^\circ \sin 75^\circ &= \frac{1}{2}\left(\sin(75^\circ + 15^\circ) - \sin(75^\circ - 15^\circ)\right) \\
&= \frac{1}{2}\left(\sin 90^\circ - \sin 60^\circ\right) \\
&= \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right) \\
&= \frac{2-\sqrt{3}}{4}
\end{aligned}$$

6. In the diagram below, points P and Q lie on the circle with radius of 2 units and centre at the origin.

   Prove $\cos(\theta - \beta) = \cos\theta\cos\beta + \sin\theta\sin\beta$.



Solution:

We can express the coordinates of P and Q in terms of the angles θ and β:

$$\text{For } Q: \quad \sin\beta = \frac{y}{2}$$
$$\therefore y = 2\sin\beta$$
$$\text{and } x = 2\cos\beta$$
$$\therefore Q(2\cos\beta; 2\sin\beta)$$

$$\text{Similarly, } P(2\cos\theta; 2\sin\theta)$$

We use the distance formula to determine PQ^2:

$$\begin{aligned}
&\text{In } \triangle POQ, \\
P\hat{O}Q &= \theta - \beta \\
d^2 &= (x_P - x_Q)^2 + (y_P - y_Q)^2 \\
PQ^2 &= (2\cos\theta - 2\cos\beta)^2 + (2\sin\theta - 2\sin\beta)^2 \\
&= 4\cos^2\theta - 8\cos\theta\cos\beta + 4\cos^2\beta + 4\sin^2\theta - 8\sin\theta\sin\beta + 4\sin^2\beta \\
&= 4\left(\cos^2\theta + \sin^2\theta\right) + 4\left(\cos^2\beta + \sin^2\beta\right) - 8\cos\theta\cos\beta - 8\sin\theta\sin\beta \\
&= 8 - 8\left(\cos\theta\cos\beta + \sin\theta\sin\beta\right)
\end{aligned}$$

Now we determine PQ^2 using the cosine rule for $\triangle POQ$:

$$\begin{aligned}
PQ^2 &= 2^2 + 2^2 - 2\,(2)\,(2)\cos(\theta - \beta) \\
&= 8 - 8\cos(\theta - \beta)
\end{aligned}$$

Equating the two expressions for PQ^2, we have

$$\begin{aligned}
8 - 8\cos(\theta - \beta) &= 8 - 8\left(\cos\theta\cos\beta + \sin\theta\sin\beta\right) \\
\therefore \cos(\theta - \beta) &= \cos\theta\cos\beta + \sin\theta\sin\beta
\end{aligned}$$

Note: earlier in this chapter we derived the compound angle identities using a unit circle (radius = 1 unit) because it simplified the calculations. From the exercise above, we see that the compound angle identities can in fact be derived using a radius of any length.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28N3	2a. 28N4	2b. 28N5	2c. 28N6	2d. 28N7	2e. 28N8
2f. 28N9	2g. 28NB	2h. 28NC	3. 28ND	4. 28NF	5. 28NG
6. 28NH					

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4.3 Double angle identities

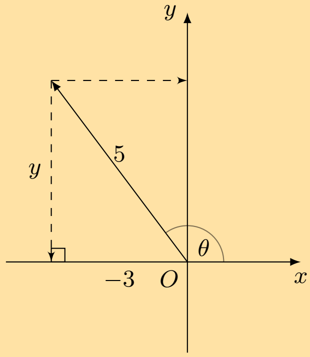
Exercise 4 – 3: Double angle identities

1. Given $5\cos\theta = -3$ and $\theta < 180°$. Determine the value of the following, without a calculator:

   a) $\cos 2\theta$

   Solution:

   Draw a sketch:



$$\begin{aligned}
\cos\theta &= -\frac{3}{5} \\
\therefore x &= -3 \qquad (\theta > 180^\circ) \\
r &= 5 \\
y^2 &= r^2 - x^2 \qquad \text{(Pythagoras)} \\
&= 5^2 - (-3)^2 \\
&= 16 \\
\therefore y &= \pm 4 \qquad \text{(but } y \text{ is positive)} \\
\therefore y &= 4
\end{aligned}$$

$$\begin{aligned}
\cos 2\theta &= 2\cos^2\theta - 1 \\
&= 2\left(-\frac{3}{5}\right)^2 - 1 \\
&= \frac{18}{25} - 1 \\
&= -\frac{7}{25}
\end{aligned}$$

b) $\sin(180^\circ - 2\theta)$

Solution:

$$\begin{aligned}
\sin(180^\circ - 2\theta) &= \sin 2\theta \\
&= 2\sin\theta\cos\theta \\
&= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) \\
&= -\frac{24}{25}
\end{aligned}$$

c) $\tan 2\theta$

Solution:

$$\begin{aligned}
\tan 2\theta &= \frac{\sin 2\theta}{\cos 2\theta} \\
&= -\frac{24}{25} \times \left(-\frac{25}{7}\right) \\
&= \frac{24}{7}
\end{aligned}$$

2. Given $\cos 40^\circ = t$, determine (without a calculator):

a) $\cos 140^\circ$

Solution:

$$\begin{aligned} \cos 140^\circ &= \cos (180^\circ - 40^\circ) \\ &= -\cos 40^\circ \\ &= -t \end{aligned}$$

b) $\sin 40^\circ$

Solution:

$$\begin{aligned} \sin 40^\circ &= \sqrt{1 - \cos^2 40^\circ} \\ &= \sqrt{1 - t^2} \end{aligned}$$

c) $\sin 50^\circ$

Solution:

$$\begin{aligned} \sin 50^\circ &= \sin (90^\circ - 50^\circ) \\ &= \cos 40^\circ \\ &= t \end{aligned}$$

d) $\cos 80^\circ$

Solution:

$$\begin{aligned} \cos 80^\circ &= \cos 2 (40^\circ) \\ &= 2\cos^2 40^\circ - 1 \\ &= 2t^2 - 1 \end{aligned}$$

e) $\cos 860^\circ$

Solution:

$$\begin{aligned} \cos 860^\circ &= \cos [2(360^\circ) + 140^\circ] \\ &= \cos 140^\circ \\ &= \cos(180^\circ - 40^\circ) \\ &= -\cos 40^\circ \\ &= -t \end{aligned}$$

f) $\cos(-1160^\circ)$

Solution:

$$\begin{aligned} \cos(-1160^\circ) &= \cos 1160^\circ \\ &= \cos (3(360^\circ) + 80^\circ) \\ &= \cos(80^\circ) \\ &= \cos 2(40^\circ) \\ &= 2\cos^2 40^\circ - 1 \\ &= 2t^2 - 1 \end{aligned}$$

3. a) Prove the identity: $\frac{1}{\sin 2A} - \frac{1}{\tan 2A} = \tan A$

Solution:

$$\begin{aligned}
\text{LHS} &= \frac{1}{\sin 2A} - \frac{1}{\tan 2A} \\
&= \frac{1}{\sin 2A} - \frac{\cos 2A}{\sin 2A} \\
&= \frac{1 - \left(1 - 2\sin^2 A\right)}{\sin 2A} \\
&= \frac{2\sin^2 A}{2\sin A \cos A} \\
&= \frac{\sin A}{\cos A} \\
&= \tan A \\
&= \text{RHS}
\end{aligned}$$

Restrictions:

$$\begin{aligned}
\sin 2A &\neq 0 \\
\therefore 2A &\neq 0^\circ + k\,.\,180^\circ \\
\therefore A &\neq 0^\circ + k\,.\,90^\circ, \quad k \in \mathbb{Z} \\
\text{And } \tan 2A &\neq 0 \\
\therefore 2A &\neq 90^\circ + k\,.\,180^\circ \\
\therefore A &\neq 45^\circ + k\,.\,90^\circ, \quad k \in \mathbb{Z} \\
\text{And for } \tan A &: \\
A &\neq 90^\circ + k\,.\,180^\circ, \quad k \in \mathbb{Z}
\end{aligned}$$

b) Hence, solve the equation $\frac{1}{\sin 2A} - \frac{1}{\tan 2A} = 0{,}75$ for $0^\circ < A < 360^\circ$.

Solution:

$$\begin{aligned}
\frac{1}{\sin 2A} - \frac{1}{\tan 2A} &= 0{,}75 \\
\therefore \tan A &= 0{,}75 \\
\therefore A &= 36{,}87^\circ \\
\text{or } A &= 180^\circ + 36{,}87^\circ \\
&= 216{,}87^\circ
\end{aligned}$$

4. Without using a calculator, find the value of the following:

a) $\sin 22{,}5^\circ$

Solution:

$$\begin{aligned}
2 \times 22{,}5^\circ &= 45^\circ \\
\cos 2\theta &= 1 - 2\sin^2\theta \\
\cos 45^\circ &= 1 - 2\sin^2(22{,}5^\circ) \\
\frac{1}{\sqrt{2}} &= 1 - 2\sin^2(22{,}5^\circ) \\
\frac{1}{\sqrt{2}} - 1 &= -2\sin^2(22{,}5^\circ) \\
\frac{1-\sqrt{2}}{\sqrt{2}} &= -2\sin^2(22{,}5^\circ) \\
\frac{1-\sqrt{2}}{-2\sqrt{2}} &= \sin^2(22{,}5^\circ) \\
\frac{\sqrt{2}-1}{2\sqrt{2}} &= \sin^2(22{,}5^\circ) \\
\therefore \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} &= \sin 22{,}5^\circ \\
\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}} &= \sin 22{,}5^\circ \\
\sqrt{\frac{2-\sqrt{2}}{4}} &= \sin 22{,}5^\circ
\end{aligned}$$

Check the answer using a calculator.

b) $\cos 67{,}5^\circ$

Solution:

$$\begin{aligned}
\cos 67{,}5^\circ &= \cos(90^\circ - 22{,}5^\circ) \\
&= \sin(22{,}5^\circ) \\
&= \sqrt{\frac{2-\sqrt{2}}{4}}
\end{aligned}$$

Check the answer using a calculator.

5. a) Prove the identity: $\tan 2x + \frac{1}{\cos 2x} = \frac{\sin x + \cos x}{\cos x - \sin x}$

Solution:

$$\begin{aligned}
\text{LHS} &= \tan 2x + \frac{1}{\cos 2x} \\
&= \frac{\sin 2x}{\cos 2x} + \frac{1}{\cos 2x} \\
&= \frac{\sin 2x + 1}{\cos 2x} \\
&= \frac{2\sin x\cos x + \cos^2 x + \sin^2 x}{\cos^2 x - \sin^2 x} \\
&= \frac{(\cos x + \sin x)^2}{(\cos x - \sin x)(\cos x + \sin x)} \\
&= \frac{\cos x + \sin x}{\cos x - \sin x} \\
&= \text{RHS}
\end{aligned}$$

Restrictions:

$$
\begin{aligned}
\cos 2x &\neq 0 \\
\therefore 2x &\neq 90^\circ + k\ .\ 180^\circ \\
\therefore x &\neq 45^\circ + k\ .\ 90^\circ, \quad k \in \mathbb{Z} \\
\text{And } \cos x &\neq \sin x \\
\therefore x &\neq 45^\circ + k\ .\ 180^\circ, \quad k \in \mathbb{Z} \\
\text{And for } \tan 2x & \\
\therefore 2x &\neq 90^\circ + k\ .\ 180^\circ \\
\therefore x &\neq 45^\circ + k\ .\ 90^\circ, \quad k \in \mathbb{Z}
\end{aligned}
$$

b) Explain why the identity is undefined for $x = 45^\circ$

Solution:

Consider the denominator on the LHS:

$$
\begin{aligned}
\cos 2x &= \cos 2\,(45^\circ) \\
&= \cos 90^\circ \\
&= 0
\end{aligned}
$$

Consider the denominator on the RHS:

$$
\begin{aligned}
\cos 45^\circ &= \sin 45^\circ \\
\therefore \cos 45^\circ - \sin 45^\circ &= 0
\end{aligned}
$$

Therefore, the identity will be undefined because division by zero is not permitted.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28NJ 2. 28NK 3. 28NM 4a. 28NN 4b. 28NP 5. 28NQ

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4.4 Solving equations

Exercise 4 – 4: Solving trigonometric equations

1. Find the general solution for each of the following equations (correct to two decimal places):

   a) $\sin 2x = \tan 28^\circ$

   Solution:

$$
\begin{aligned}
\sin 2x &= \tan 28^\circ \\
&= 0{,}53\ldots \\
\text{ref } \angle &= 32{,}12^\circ \\
\text{First quadrant:} \quad 2x &= 32{,}12^\circ + k\ .\ 360^\circ \\
\therefore x &= 16{,}06^\circ + k\ .\ 180^\circ, k \in \mathbb{Z} \\
\\
\text{Second quadrant:} \quad 2x &= (180^\circ - 32{,}12^\circ) + k\ .\ 360^\circ \\
&= 147{,}88^\circ + k\ .\ 360^\circ \\
\therefore x &= 73{,}9^\circ + k\ .\ 180^\circ, \quad k \in \mathbb{Z}
\end{aligned}
$$

b) $\cos y = \sin 2y$

Solution:

We work with the left-hand side of the equation since it contains y, which is easier to simplify than the double angle on the right-hand side:

$$\begin{aligned}
\cos y &= \sin 2y \\
\sin(90^\circ - y) &= \sin 2y \\
\text{First quadrant:}\quad (90^\circ - y) + k\,.\,360^\circ &= 2y \\
90^\circ + k\,.\,360^\circ &= 3y \\
30^\circ + k\,.\,120^\circ &= y
\end{aligned}$$

$$\begin{aligned}
\text{Second quadrant:}\quad 180^\circ - (90^\circ - y) + k\,.\,360^\circ &= 2y \\
90^\circ + y + k\,.\,360^\circ &= 2y \\
90^\circ + k\,.\,360^\circ &= y
\end{aligned}$$

for $k \in \mathbb{Z}$.

c) $\sin 2\alpha = \cos 2\alpha$

Solution:

$$\begin{aligned}
\sin 2\alpha &= \cos 2\alpha \\
\frac{\sin 2\alpha}{\cos 2\alpha} &= 1 \\
\tan 2\alpha &= 1 \\
\therefore 2\alpha &= 45^\circ + k\,.\,180^\circ \\
\therefore \alpha &= 22{,}5^\circ + k\,.\,90^\circ, k \in \mathbb{Z}
\end{aligned}$$

d) $\sin 3p = \sin 2p$

Solution:

$$\begin{aligned}
\sin 3p &= \sin 2p \\
\text{First quadrant:}\quad 3p &= 2p + k\,.\,360^\circ \\
\therefore p &= k\,.\,360^\circ \\
\text{Second quadrant:}\quad 3p &= (180^\circ - 2p) + k\,.\,360^\circ \\
5p &= 180^\circ + k\,.\,360^\circ \\
\therefore p &= 36^\circ + k\,.\,72^\circ, k \in \mathbb{Z}
\end{aligned}$$

e) $\tan A = \frac{1}{\tan A}$

Solution:

$$\begin{aligned}
\tan A &= \frac{1}{\tan A} \\
\tan^2 A &= 1 \\
\therefore \tan A &= \pm 1 \\
\text{First quadrant:}\quad A &= 45^\circ + k\,.\,180^\circ \\
\text{Second quadrant:}\quad A &= (180^\circ - 45^\circ) + k\,.\,180^\circ \\
&= 135^\circ + k\,.\,180^\circ, k \in \mathbb{Z}
\end{aligned}$$

Restrictions:

$$\begin{aligned}
&\text{For } \tan A: \\
&\quad A \neq 90^\circ + k\,.\,180^\circ, k \in \mathbb{Z} \\
&\text{And for } \frac{1}{\tan A} \text{ we can write } \frac{\cos A}{\sin A}: \\
&\therefore \sin A \neq 0 \\
&\quad A \neq 0^\circ + k\,.\,180^\circ, k \in \mathbb{Z}
\end{aligned}$$

f) $\sin x \tan x = 1$

Solution:

$$\begin{aligned}
\sin x \tan x &= 1 \\
\sin x \ . \ \frac{\sin x}{\cos x} &= 1 \qquad (\cos x \neq 0) \\
\frac{\sin^2 x}{\cos x} &= 1 \\
1 - \cos^2 x &= \cos x \\
\cos^2 x + \cos x - 1 &= 0 \\
\text{Let } \cos x &= p \\
p^2 + p - 1 &= 0 \\
\therefore p &= \frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)} \qquad \text{(quadratic formula)} \\
&= \frac{-1 \pm \sqrt{5}}{2} \\
\therefore p = \frac{-1 - \sqrt{5}}{2} &\text{ or } p = \frac{-1 + \sqrt{5}}{2} \\
\therefore \cos x = \frac{-1 - \sqrt{5}}{2} &\text{ or } \cos x = \frac{-1 + \sqrt{5}}{2} \\
\cos x = -1{,}618\ldots &\text{ or } \cos x = 0{,}618\ldots \\
\text{First answer: no solution since } &-1 \leq \cos x \leq 1 \\
\text{Second answer: ref } \angle &= 51{,}8^\circ \\
\text{First quadrant: } \quad x &= 51{,}8^\circ + k \ . \ 360^\circ \\
\text{Second quadrant: } \quad x &= (360^\circ - 51{,}8^\circ) + k \ . \ 360^\circ \\
&= 308{,}2^\circ + k \ . \ 360^\circ, k \in \mathbb{Z}
\end{aligned}$$

Restrictions:

$$\begin{aligned}
&\text{For } \tan x: \\
&\quad x \neq 90^\circ + k \ . \ 180^\circ, \quad k \in \mathbb{Z}
\end{aligned}$$

g) $\sin t \ . \ \sin 2t + \cos 2t = 1$

Solution:

$$\begin{aligned}
\sin t \ . \ \sin 2t + \cos 2t &= 1 \\
\sin t \ . \ 2 \sin t \cos t + \left(1 - 2\sin^2 t\right) &= 1 \\
2\sin^2 t \cos t + 1 - 2\sin^2 t &= 1 \\
2\sin^2 t \cos t - 2\sin^2 t &= 0 \\
2\sin^2 t \left(\cos t - 1\right) &= 0 \\
\text{If } \quad 2\sin^2 t &= 0 \\
\sin t &= 0 \\
\therefore t &= 0^\circ + k \ . \ 360^\circ \\
\text{or } \quad t &= 180^\circ + k \ . \ 360^\circ \\
\text{If } \quad \cos t - 1 &= 0 \\
\cos t &= 1 \\
\therefore t &= 0^\circ + k \ . \ 360^\circ \\
\text{or } \quad t &= 360^\circ + k \ . \ 360^\circ
\end{aligned}$$

Final answer: $t = 0^\circ + k \ . \ 180^\circ, \quad k \in \mathbb{Z}$

h) $\sin 60^\circ \cos x + \cos 60^\circ \sin x = 1$

Solution:

$$\begin{aligned}
\sin 60^\circ \cos x + \cos 60^\circ \sin x &= 1 \\
\sin(60^\circ + x) &= 1 \\
\therefore 60^\circ + x &= 90^\circ + k\ .\ 360^\circ, \quad k \in \mathbb{Z} \\
\therefore x &= 30^\circ + k\ .\ 360^\circ
\end{aligned}$$

2. Given: $\sin x \cos x = \sqrt{3}\sin^2 x$

a) Solve the equation for $x \in [0^\circ; 360^\circ]$, without using a calculator.

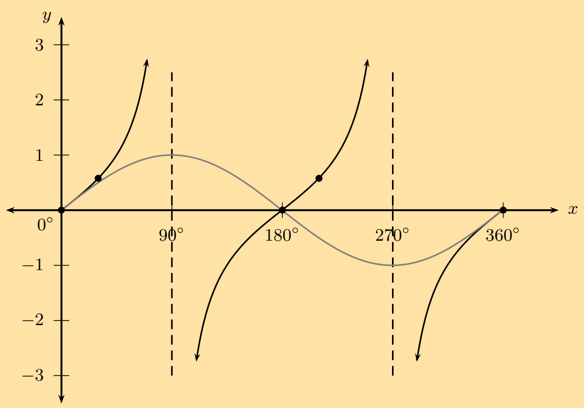
Solution:

$$\begin{aligned}
\sin x \cos x &= \sqrt{3}\sin^2 x \\
\sin x \cos x - \sqrt{3}\sin^2 x &= 0 \\
\sin x \left(\cos x - \sqrt{3}\sin x\right) &= 0 \\
\therefore \sin x = 0 \text{ or } \cos x - \sqrt{3}\sin x &= 0 \\
\text{If } \sin x = 0 &\qquad \text{for } x \in [0^\circ; 360^\circ] \\
\therefore x &= 0^\circ, 180^\circ \text{ or } 360^\circ \\
\text{If } \cos x - \sqrt{3}\sin x = 0 &\qquad \text{for } x \in [0^\circ; 360^\circ] \\
\cos x &= \sqrt{3}\sin x \\
\frac{\cos x}{\cos x} &= \sqrt{3}\frac{\sin x}{\cos x} \\
1 &= \sqrt{3}\tan x \\
\frac{1}{\sqrt{3}} &= \tan x \\
\text{ref } \angle &= 30^\circ \\
\therefore x &= 30^\circ + k\ .\ 180^\circ, k \in \mathbb{Z} \\
\therefore x &= 30^\circ \text{ or } 210^\circ
\end{aligned}$$

b) Draw a graph and indicate the solution on the diagram.

Solution:

The diagram below shows the graph of $y = \sin x$ (grey) and $y = \tan x$ (black).



3. Given: $1 + \tan^2 2A = 5\tan 2A - 5$

a) Determine the general solution.

Solution:

$$1 + \tan^2 2A = 5\tan 2A - 5$$
$$\tan^2 2A - 5\tan 2A + 6 = 0$$
$$(\tan 2A - 3)(\tan 2A - 2) = 0$$
$$\therefore \tan 2A - 3 = 0 \text{ or } \tan 2A - 2 = 0$$
$$\text{If } \tan 2A - 3 = 0$$
$$\tan 2A = 3$$
$$\therefore 2A = 71{,}57^\circ + k \;.\; 180^\circ$$
$$\therefore A = 35{,}79^\circ + k \;.\; 90^\circ, \quad k \in \mathbb{Z}$$
$$\text{If } \tan 2A - 2 = 0$$
$$\tan 2A = 2$$
$$\therefore 2A = 63{,}43^\circ + k \;.\; 180^\circ$$
$$\therefore A = 31{,}72^\circ + k \;.\; 90^\circ, k \in \mathbb{Z}$$

b) How many solutions does the given equation have in the interval $[-90^\circ; 360^\circ]$?

Solution:

$$\text{If } k = -1: \quad A = 35{,}79^\circ - 90^\circ$$
$$= -54{,}21^\circ$$
$$A = 31{,}72^\circ - 90^\circ$$
$$= -58{,}28^\circ$$
$$\text{If } k = 0: \quad A = 35{,}79^\circ$$
$$A = 31{,}72^\circ$$
$$\text{If } k = 1: \quad A = 35{,}79^\circ + 90^\circ$$
$$= 125{,}79^\circ$$
$$A = 31{,}72^\circ + 90^\circ$$
$$= 121{,}72^\circ$$
$$\text{If } k = 2: \quad A = 35{,}79^\circ + 180^\circ$$
$$= 215{,}79^\circ$$
$$A = 31{,}72^\circ + 180^\circ$$
$$= 211{,}72^\circ$$
$$\text{If } k = 3: \quad A = 35{,}79^\circ + 270^\circ$$
$$= 305{,}79^\circ$$
$$A = 31{,}72^\circ + 270^\circ$$
$$= 301{,}72^\circ$$

Therefore, there are ten solutions that lie within the interval $[-90^\circ; 360^\circ]$.

4. Without using a calculator, solve $\cos(A - 25^\circ) + \cos(A + 25^\circ) = \cos 25^\circ$ in $[-360^\circ; 360^\circ]$.

   Solution:

   First we simplify the left-hand side of the equation:

$$\cos(A - 25^\circ) + \cos(A + 25^\circ) = \cos 25^\circ$$
$$\cos A \cos 25^\circ + \sin A \sin 25^\circ + \cos A \cos 25^\circ - \sin A \sin 25^\circ = \cos 25^\circ$$
$$2\cos A \cos 25^\circ = \cos 25^\circ$$
$$\therefore 2\cos A = 1 \quad (\cos 25^\circ \neq 0)$$
$$\cos A = \frac{1}{2}$$
$$\therefore A = 60^\circ + k \;.\; 360^\circ$$
$$\text{or } A = 300^\circ + k \;.\; 360^\circ, \quad k \in \mathbb{Z}$$
$$\therefore A = -300^\circ, -60^\circ, 60^\circ \text{ or } 300^\circ$$

5. a) Find the general solution for $\sin x \cos 3x + \cos x \sin 3x = \tan 140°$.

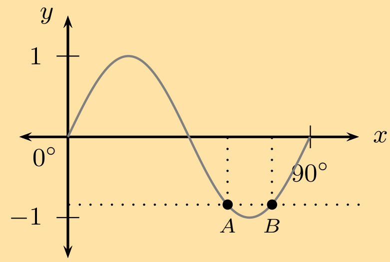
Solution:

$$\begin{aligned}
\sin x \cos 3x + \cos x \sin 3x &= \tan 140° \\
\sin(x + 3x) &= -0{,}839\ldots \\
\sin 4x &= -0{,}839\ldots \\
\text{ref } \angle &= 57{,}03° \\
\text{Third quadrant: } \quad 4x &= (180° + 57{,}03°) + k \,.\, 360°, k \in \mathbb{Z} \\
&= 237{,}03° + k \,.\, 360° \\
\therefore x &= 59{,}26° + k \,.\, 90° \\
\text{Fourth quadrant: } \quad 4x &= (360° - 57{,}03°) + k \,.\, 360°, k \in \mathbb{Z} \\
&= 302{,}97° + k \,.\, 360° \\
\therefore x &= 75{,}74° + k \,.\, 90°
\end{aligned}$$

b) Use a graph to illustrate the solution for the interval $[0°; 90°]$.

Solution:

The diagram below shows the graph of $y = \sin 4x$.

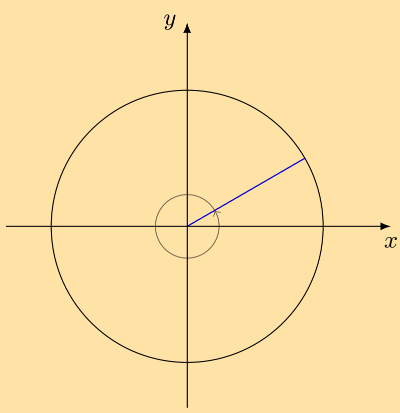


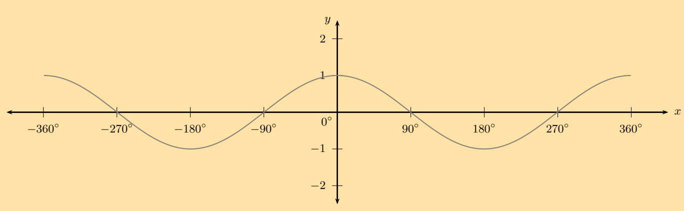
$A(59{,}26°; -0{,}84), B(75{,}74°; -0{,}84)$

6. Explain why the general solution for the equation $\cos\theta = a$ is $\theta = \cos^{-1} a + k \,.\, 360°$ and the general solution for $\tan\theta = a$ is $\theta = \tan^{-1} a + k \,.\, 180°$. Why are they different?

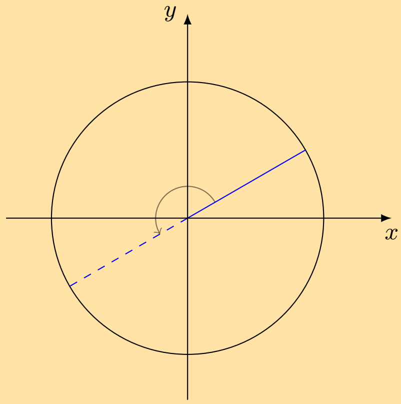
Solution:

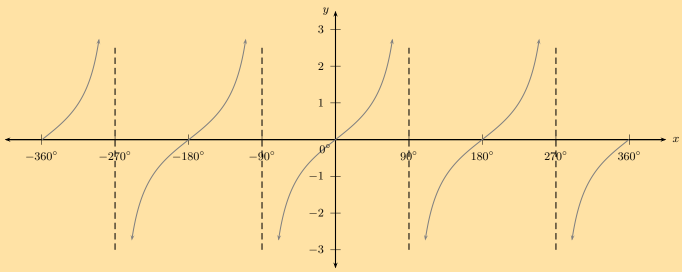
The period of the cosine function is $360°$. This means that the function values repeat after $360°$.





The period of the tangent function is $180°$. This means that the function values repeat after $180°$.





7. Solve for x: $\sqrt{3}\sin x + \cos x = 2$

Solution:

$$\begin{aligned}
\sqrt{3}\sin x + \cos x &= 2 \\
\sqrt{3}\sin x - 2 &= \cos x \\
\text{Substitute } \cos x &= \sqrt{1-\sin^2 x} \\
\therefore \sqrt{3}\sin x - 2 &= \sqrt{1-\sin^2 x} \\
\text{Square both sides of eqn: } \left(\sqrt{3}\sin x - 2\right)^2 &= \left(\sqrt{1-\sin^2 x}\right)^2 \\
3\sin^2 x - 4\sqrt{3}\sin x + 4 &= 1 - \sin^2 x \\
4\sin^2 x - 4\sqrt{3}\sin x + 3 &= 0 \\
\left(2\sin x - \sqrt{3}\right)^2 &= 0 \\
\therefore 2\sin x - \sqrt{3} &= 0 \\
\therefore \sin x &= \frac{\sqrt{3}}{2} \\
\text{ref } \angle &= 60° \\
\text{First quadrant: } x &= 60° + k\ .\ 360°, \quad k \in \mathbb{Z} \\
\text{Second quadrant: } x &= 180° - 60° + k\ .\ 360° \\
&= 120° + k\ .\ 360°, \quad k \in \mathbb{Z}
\end{aligned}$$

Check that both answers satisfy the original equation:

Substitute $x = 60^\circ$

$$\therefore \text{LHS} = \sqrt{3}\sin 60^\circ + \cos 60^\circ$$
$$= \sqrt{3}\ .\ \frac{\sqrt{3}}{2} + \frac{1}{2}$$
$$= \frac{3}{2} + \frac{1}{2}$$
$$= 2$$
$$= \text{RHS}$$

Substitute $x = 120^\circ$

$$\therefore \text{LHS} = \sqrt{3}\sin 120^\circ + \cos 120^\circ$$
$$= \sqrt{3}\ .\ \frac{\sqrt{3}}{2} - \frac{1}{2}$$
$$= \frac{3}{2} - \frac{1}{2}$$
$$= 1$$
$$\neq \text{RHS}$$

Therefore, $x = 60^\circ + k\ .\ 360^\circ$.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28NR 1b. 28NS 1c. 28NT 1d. 28NV 1e. 28NW 1f. 28NX
1g. 28NY 1h. 28NZ 2. 28P2 3. 28P3 4. 28P4 5. 28P5
6. 28P6 7. 28P7

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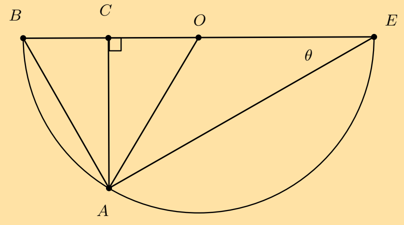
4.5 Applications of trigonometric functions

Problems in two dimensions

Exercise 4 – 5:

1. In the diagram below, O is the centre of the semi-circle BAE.



a) Find $A\hat{O}C$ in terms of θ.

Solution:

BE is a diameter of semi-circle BAE. O is the centre and therefore bisects BE.

$OA = OE = OB$ (equal radii)

$\therefore O\hat{A}E = \theta$ (opp. $\angle$'s of equal sides)

$\therefore A\hat{O}C = \theta + \theta$ (ext. $\angle$ of $\triangle OAE$ = sum int. opp.)

$= 2\theta$

Or

$$A\hat{O}C = 2\theta \quad (\angle \text{ at centre } = 2\angle \text{ at circum. })$$

b) In $\triangle ABE$, determine an expression for $\cos\theta$.

Solution:

$$B\hat{A}E = 90^\circ \quad (\angle \text{ in semi circle})$$
$$\therefore \cos\theta = \frac{AE}{BE}$$

c) In $\triangle ACE$, determine an expression for $\sin\theta$.

Solution:

$$A\hat{C}E = 90^\circ \quad (\text{given})$$
$$\therefore \sin\theta = \frac{CA}{AE}$$

d) In $\triangle ACO$, determine an expression for $\sin 2\theta$.

Solution:

$$A\hat{C}O = 90^\circ \quad (\text{given})$$
$$A\hat{O}C = 2\theta \quad (\text{proven})$$
$$\therefore \sin 2\theta = \frac{CA}{AO}$$

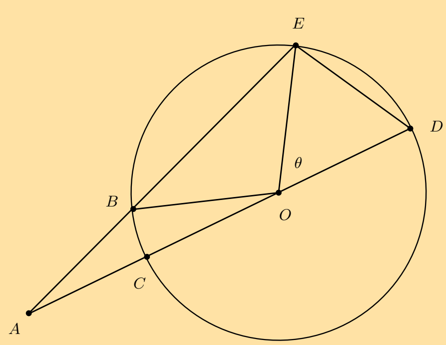
e) Use the results from the previous questions to show that $\sin 2\theta = 2\sin\theta\cos\theta$.

Solution:

$$\begin{aligned}
\cos\theta &= \tfrac{AE}{BE} \\
\therefore AE &= BE\cos\theta \\
\sin\theta &= \tfrac{CA}{AE} \\
\therefore CA &= AE\sin\theta \\
&= (BE\cos\theta)\sin\theta \\
BE &= 2(OE) && \text{(radii)} \\
\therefore CA &= 2(OE)\sin\theta\cos\theta \\
\therefore \tfrac{CA}{OE} &= 2\sin\theta\cos\theta \\
\text{but } \tfrac{CA}{OE} &= \tfrac{CA}{OA} = \sin 2\theta && (\text{in } \triangle ACO) \\
\therefore \sin 2\theta &= 2\sin\theta\cos\theta
\end{aligned}$$

2. DC is a diameter of the circle with centre O and radius r. $CA = r$, $AE = 2DE$ and $D\hat{O}E = \theta$.

   Show that $\cos\theta = \frac{1}{4}$.



Solution:

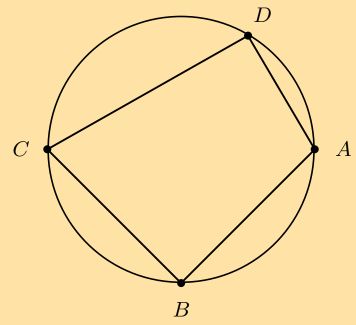
In $\triangle DOE$, let $DE = k$

$$k^2 = r^2 + r^2 - 2r^2 \cos\theta$$
$$= 2r^2 - 2r^2 \cos\theta \ldots\ldots (1)$$

In $\triangle AOE$, $AE = 2k$

$$(2k)^2 = (2r)^2 + r^2 - 2(2r)(r)\cos(180^\circ - \theta)$$
$$\therefore 4k^2 = 4r^2 + r^2 + 4r^2 \cos\theta$$
$$\therefore 4k^2 = 5r^2 + 4r^2 \cos\theta \ldots\ldots (2)$$
$$(1) \times 4:\ 4k^2 = 8r^2 - 8r^2 \cos\theta \ldots\ldots (3)$$
$$(3) - (2):\quad 0 = 3r^2 - 12r^2 \cos\theta$$
$$\therefore 12r^2 \cos\theta = 3r^2$$
$$\cos\theta = \frac{3r^2}{12r^2}$$
$$\therefore \cos\theta = \frac{1}{4}$$

3. The figure below shows a cyclic quadrilateral with $\frac{BC}{CD} = \frac{AD}{AB}$.



a) Show that the area of the cyclic quadrilateral is $DC\ .\ DA\ .\ \sin\hat{D}$.

Solution:

Connect CA to construct $\triangle ADC$ and $\triangle ABC$

$$\frac{BC}{CD} = \frac{AD}{AB} \quad \text{(given)}$$
$$\therefore BC\ .\ AB = AD\ .\ CD$$
$$\text{Area } \triangle ADC = \tfrac{1}{2} DC\ .\ DA \sin\hat{D}$$
$$\text{Area } \triangle ABC = \tfrac{1}{2} AB\ .\ BC \sin\hat{B}$$
$$\hat{B} + \hat{D} = 180^\circ \quad \text{(opp. } \angle\text{s cyclic quad. supp.)}$$
$$\therefore \hat{B} = 180^\circ - \hat{D}$$
$$\therefore \sin\hat{B} = \sin(180^\circ - \hat{D}) = \sin\hat{D}$$
$$\text{Area } ABCD = \text{area } \triangle ADC + \text{area } \triangle ABC$$
$$= \tfrac{1}{2} DC\ .\ DA \sin\hat{D} + \tfrac{1}{2} AB\ .\ BC \sin\hat{B}$$
$$= \tfrac{1}{2} DC\ .\ DA \sin\hat{D} + \tfrac{1}{2} DC\ .\ DA \sin\hat{D}$$
$$= DC\ .\ DA \sin\hat{D}$$

b) Write down two expressions for CA^2: one in terms of $\cos\hat{D}$ and one in terms of $\cos\hat{B}$.

Solution:

$$CA^2 = DC^2 + DA^2 - 2(DC)(DA)\cos\hat{D}$$
$$CA^2 = AB^2 + BC^2 - 2(AB)(BC)\cos\hat{B}$$

c) Show that $2CA^2 = CD^2 + DA^2 + AB^2 + BC^2$.

Solution:

$$\begin{aligned}
CA^2 &= DC^2 + DA^2 - 2(DC)(DA)\cos\hat{D} \\
CA^2 &= AB^2 + BC^2 - 2(AB)(BC)\cos\hat{B} \\
&= AB^2 + BC^2 - 2(DC)(DA)\cos\hat{B} \quad (DC\ .\ DA = AB\ .\ BC) \\
2CA^2 &= CD^2 + DA^2 + AB^2 + BC^2 - 2(DC)(DA)\cos\hat{D} \\
&- 2(DC)(DA)\cos(180^\circ - \hat{D}) \\
&= CD^2 + DA^2 + AB^2 + BC^2 - 2(DC)(DA)\cos\hat{D} \\
&+ 2(DC)(DA)\cos\hat{D} \\
&= CD^2 + DA^2 + AB^2 + BC^2
\end{aligned}$$

d) Suppose that $BC = 10$ units, $CD = 15$ units, $AD = 4$ units and $AB = 6$ units. Calculate CA^2 (correct to one decimal place).

Solution:

$$\begin{aligned}
2CA^2 &= (15)^2 + (4)^2 + (6)^2 + (10)^2 \\
&= 377 \\
\therefore CA^2 &= 188{,}5 \text{ units}^2
\end{aligned}$$

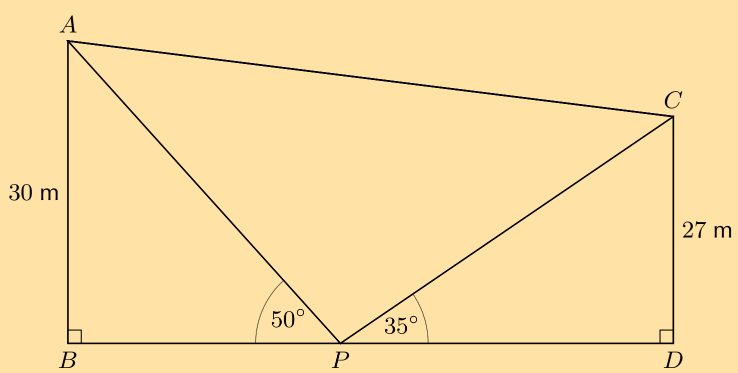
e) Find the angle $\hat{B}$. Hence, calculate the area of $ABCD$ (correct to one decimal place).

Solution:

$$\begin{aligned}
CA^2 &= AB^2 + BC^2 - 2(AB)(BC)\cos\hat{B} \\
188{,}5 &= 6^2 + 10^2 - 2(6)(10)\cos\hat{B} \\
188{,}5 &= 136 - 120\cos\hat{B} \\
\cos\hat{B} &= -0{,}4375 \\
\text{ref } \angle &= 64{,}05^\circ \\
\therefore \hat{B} &= 180^\circ - 64{,}05^\circ \\
&= 115{,}94^\circ
\end{aligned}$$

$$\begin{aligned}
\text{Area } ABCD &= DC \times DA\sin(115{,}94^\circ) \\
&= 15 \times 4\sin(115{,}94^\circ) \\
&= \sin(115{,}94^\circ) \\
&= 54{,}0 \text{ units}^2
\end{aligned}$$

4. Two vertical towers AB and CD are 30 m and 27 m high, respectively. Point P lies between the two towers. The angle of elevation from P to A is 50° and from P to C is 35°. A cable is needed to connect A and C.



a) Determine the minimum length of cable needed to connect A and C (to the nearest metre).

Solution:

$$\hat{B} = \hat{D} = 90^\circ \quad \text{(vertical towers)}$$

$$\begin{aligned} \text{In } \triangle ABP: \quad \frac{30}{AP} &= \sin 50^\circ \\ AP &= \frac{30}{\sin 50^\circ} \\ &= 39{,}16 \text{ m} \end{aligned}$$

$$\begin{aligned} \text{In } \triangle CDP: \quad \frac{27}{CP} &= \sin 35^\circ \\ CP &= \frac{27}{\sin 35^\circ} \\ &= 47{,}07 \text{ m} \end{aligned}$$

$$\begin{aligned} \text{In } \triangle APC: A\hat{P}C &= 180^\circ - (50^\circ + 35^\circ) \\ &= 95^\circ \\ AC^2 &= AP^2 + PC^2 - 2(AP)(PC)\cos A\hat{P}C \\ &= (39{,}16)^2 + (47{,}07)^2 - 2(39{,}16)(47{,}07)\cos 95^\circ \\ &= 4070{,}39 \\ \therefore AC &= 63{,}80 \text{ m} \\ &\approx 64 \text{ m} \end{aligned}$$

b) How far apart are the bases of the two towers (to the nearest metre)?

Solution:

$$\begin{aligned} \text{In } \triangle ABP: \quad \frac{BP}{39{,}16} &= \cos 50^\circ \\ \therefore BP &= 39{,}16 \cos 50^\circ \\ &= 25{,}17 \text{ m} \end{aligned}$$

$$\begin{aligned} \text{In } \triangle CDP: \quad \frac{PD}{47{,}07} &= \cos 35^\circ \\ \therefore PD &= 47{,}07 \cos 35^\circ \\ &= 38{,}56 \text{ m} \end{aligned}$$

$$\begin{aligned} BD &= BP + PD \\ &= 25{,}17 \text{ m} + 38{,}56 \text{ m} \\ &= 63{,}7 \text{ m} \\ &\approx 64 \text{ m} \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28P8 1b. 28P9 1c. 28PB 1d. 28PC 1e. 28PD 2. 28PF
3a. 28PG 3b. 28PH 3c. 28PJ 3d. 28PK 3e. 28PM 4. 28PN

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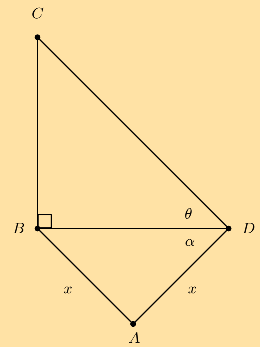
Exercise 4 – 6:

1. The line BC represents a tall tower, with B at its base. The angle of elevation from D to C is θ. A man stands at A such that $BA = AD = x$ and $A\hat{D}B = \alpha$.



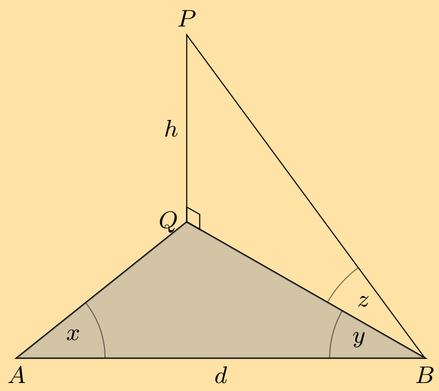
a) Find the height of the tower BC in terms of x, $\tan\theta$ and $\cos\alpha$.
Solution:

$$\begin{aligned}
\text{In } \triangle BCD, \quad \tan\theta &= \frac{BC}{BD} \\
\therefore BC &= BD\tan\theta \\
\text{In } \triangle ABD, \quad \frac{\sin\hat{A}}{BD} &= \frac{\sin\alpha}{x} \\
\therefore BD &= \frac{x\sin\hat{A}}{\sin\alpha} \\
\therefore BC &= \frac{x\sin\hat{A}\tan\theta}{\sin\alpha} \\
\hat{A} + \alpha + \alpha &= 180^\circ \qquad (\angle\text{'s in } \triangle ABD) \\
\therefore \hat{A} &= 180^\circ - 2\alpha \\
\therefore \sin\hat{A} &= \sin(180^\circ - 2\alpha) = \sin 2\alpha \\
\therefore BC &= \frac{x\sin 2\alpha\tan\theta}{\sin\alpha} \\
\text{and } \sin 2\alpha &= 2\sin\alpha\cos\alpha \\
\therefore \frac{\sin 2\alpha}{\sin\alpha} &= 2\cos\alpha \\
\therefore BC &= 2x\cos\alpha\tan\theta
\end{aligned}$$

b) Find BC if we are given that $x = 140$ m, $\alpha = 21^\circ$ and $\theta = 9^\circ$.
Solution:

$$BC = 2(140)\cos 21^\circ \tan 9^\circ = 41{,}40 \text{ m}$$

2. P is the top of a mast and its base, Q, is in the same horizontal plane as the points A and B. The angle of elevation measured from B to P is z. $AB = d$, $Q\hat{A}B = x$ and $Q\hat{B}A = y$.



a) Use the given information to derive a general formula for h, the height of the mast.

Solution:

In $\triangle QAB$: $\hat{Q} = 180° - (x + y)$ ($\angle$s sum of $\triangle QAB$)

$$\frac{QB}{\sin \hat{A}} = \frac{AB}{\sin \hat{Q}}$$
$$\frac{QB}{\sin x} = \frac{d}{\sin\left(180° - (x+y)\right)}$$
$$\frac{QB}{\sin x} = \frac{d}{\sin(x+y)}$$
$$QB = \frac{d \sin x}{\sin(x+y)}$$

In $\triangle PQB$:

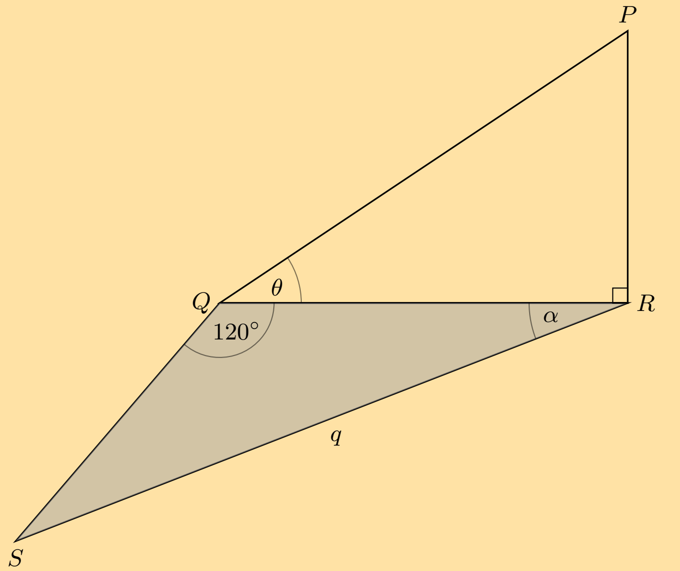
$$\hat{Q} = 90° \quad \text{(vertical mast)}$$
$$\hat{B} = z \quad \text{(given)}$$
$$\tan \hat{B} = \frac{QP}{QB}$$
$$\tan \hat{B} \times QB = QP$$
$$\therefore QP = \tan z \times QB$$
$$\therefore h = \tan z \times \frac{d \sin x}{\sin(x+y)}$$
$$= \frac{d \sin x \tan z}{\sin(x+y)}$$

b) If $d = 50$ m, $x = 46°$, $y = 15°$ and $z = 20°$, calculate h (to the nearest metre).

Solution:

$$h = \frac{d \sin x \tan z}{\sin(x+y)}$$
$$= \frac{50 \sin 46° \tan 20°}{\sin(46° + 15°)}$$
$$= 14{,}967\ldots$$
$$\approx 15 \text{ m} \quad \text{(to nearest metre)}$$

3. PR is the height of a block of flats with R at the base and P at the top of the building. S is a point in the same horizontal plane as points Q and R. $SR = q$ units, $S\hat{Q}R = 120°$, $S\hat{R}Q = \alpha$ and $R\hat{Q}P = \theta$.



a) Show that the height of the block of flats, PR, can be expressed as:

$$PR = q\tan\theta\left(\cos\alpha - \frac{\sqrt{3}\sin\alpha}{3}\right)$$

Solution:

QR is the link between $\triangle PQR$ and $\triangle SQR$.

In $\triangle SQR$: $\hat{S} = 180° - (120° + \alpha)$ ($\angle$s sum of $\triangle SQR$)

$$\frac{QR}{\sin\hat{S}} = \frac{q}{\sin\hat{Q}}$$

$$QR = \frac{q\sin(180° - (120° + \alpha))}{\sin(120°)}$$

$$= \frac{q\sin(60° - \alpha)}{\sin(180° - 60°)}$$

$$= \frac{q(\sin 60°\cos\alpha - \cos 60°\sin\alpha)}{\sin 60°}$$

$$= \frac{q\left(\frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha\right)}{\frac{\sqrt{3}}{2}}$$

$$= q\left(\frac{\sqrt{3}\cos\alpha - \sin\alpha}{2}\right) \times \frac{2}{\sqrt{3}}$$

$$= q\left(\cos\alpha - \frac{\sin\alpha}{\sqrt{3}}\right)$$

$$= q\left(\cos\alpha - \left(\frac{\sin\alpha}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\right)\right)$$

$$= q\left(\cos\alpha - \frac{\sqrt{3}\sin\alpha}{3}\right)$$

In $\triangle PQR$: $\hat{R} = 90°$

$$\frac{PR}{QR} = \tan\theta$$

$$\therefore PR = \tan\theta \times QR$$

$$= \tan\theta \times q\left(\cos\alpha - \frac{\sqrt{3}\sin\alpha}{3}\right)$$

$$= q\tan\theta\left(\cos\alpha - \frac{\sqrt{3}\sin\alpha}{3}\right)$$

b) If $SR = 35$ m, $S\hat{R}Q = 16°$ and $R\hat{Q}P = 30°$, calculate PR (correct to one decimal place).

Solution:

$$\begin{aligned} PR &= q \tan\theta \left(\cos\alpha - \frac{\sqrt{3}\sin\alpha}{3}\right) \\ &= 35\tan 30° \left(\cos 16° - \frac{\sqrt{3}\sin 16°}{3}\right) \\ &= 16{,}2 \text{ m} \end{aligned}$$

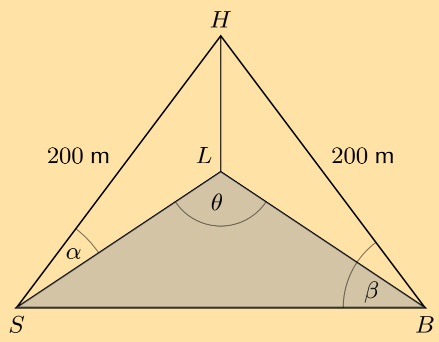
c) Assuming each level is 2,5 m high, estimate the number of levels in the block of flats.

Solution:

$$\begin{aligned} \text{Approx. no. of levels } &= \frac{16}{2{,}5} \\ &= 6{,}4 \end{aligned}$$

Therefore, there are approximately 6 levels.

4. Two ships at sea can see a lighthouse on the shore. The distance from the top of the lighthouse (H) to ship S and to ship B is 200 m. The angle of elevation from S to H is α, $H\hat{B}S = \beta$ and $S\hat{L}B = \theta$



a) Show that the distance between the two ships is given by $SB = 400\cos\beta$.

Solution:

$$\begin{aligned} &\text{In } \triangle HSB: \\ HS &= HB = 200 \text{ m} && \text{(given)} \\ SB^2 &= HS^2 + HB^2 - 2(HS)(HB)\cos S\hat{H}B && \text{(cosine rule)} \\ SB^2 &= (200)^2 + (200)^2 - 2(200)(200)\cos S\hat{H}B \\ &= 40\ 000 + 40\ 000 - 80\ 000\cos(180° - 2\beta) \\ &= 80\ 000 + 80\ 000\cos(2\beta) \\ &= 80\ 000(1 + \cos 2\beta) \\ &= 80\ 000\left(1 + 2\cos^2\beta - 1\right) \\ &= 160\ 000\cos^2\beta \\ \therefore SB &= 400\cos\beta && \text{(distance positive)} \end{aligned}$$

b) Show that the area of the sea included in $\triangle LSB$ is given by area $\triangle LSB = 2000\cos^2\alpha\sin\theta$.

Solution:

$$\begin{aligned}
\text{In } \triangle HSL \text{ and } \triangle HBL: & \\
HS = HB &= 200 \text{ m} && \text{(given)} \\
HL &= HL && \text{(common side)} \\
H\hat{L}S = H\hat{L}B &= 90^\circ && \text{(vertical lighthouse)} \\
\therefore \triangle HSL &\,|||\, \triangle HBL && \text{(RHS)} \\
\therefore H\hat{B}L = H\hat{S}L &= \alpha && (\triangle HSL|||\triangle HBL) \\
\text{Area}\triangle LSB &= \tfrac{1}{2}(LS)(LB)\sin\theta \\
\text{In } \triangle HSL: \quad \tfrac{LS}{HS} &= \cos\alpha \\
\therefore LS &= HS\cos\alpha \\
&= 200\cos\alpha \\
\text{In } \triangle HBL: \quad \tfrac{LB}{HB} &= \cos\alpha \\
\therefore LB &= 200\cos\alpha
\end{aligned}$$

$$\begin{aligned}
\text{Area}LSB &= \tfrac{1}{2}(LS)(LB)\sin\theta \\
&= \tfrac{1}{2}(200\cos\alpha)(200\cos\alpha)\sin\theta \\
&= 20\ 000\cos^2\alpha\sin\theta
\end{aligned}$$

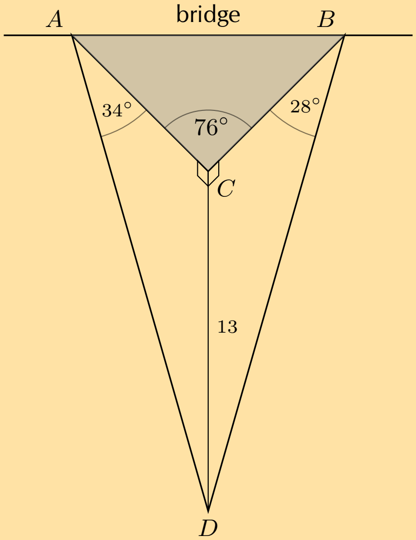
c) Calculate the triangular area of the sea if the angle of inclination from the ship to the top of the lighthouse is 10° and the angle between the direct lines from the base of the lighthouse to each ship is 85°.

Solution:

$\theta = 85^\circ$ and $\alpha = 10^\circ$.

$$\begin{aligned}
\text{Area } \triangle LSB &= 2000\cos^2\alpha\sin\theta \\
&= 2000(\cos 10^\circ)^2\sin 85^\circ \\
&= 1932{,}3 \text{ m}^2
\end{aligned}$$

5. A triangular look-out platform ($\triangle ABC$) is attached to a bridge that extends over a deep gorge. The vertical depth of the gorge, the distance from the edge of the look-out C to the bottom of the gorge D, is 13 m. The angle of depression from A to D is 34° and from B to D is 28°. The angle at the edge of the platform, $\hat{C}$ is 76°.



a) Calculate the area of the look-out platform (to the nearest m^2).

Solution:

$$\text{Area } \triangle ABC = \frac{1}{2} AC \,.\, BC \sin \hat{C}$$

$$\text{In } \triangle ACD: \quad \frac{13}{AC} = \tan 34^\circ$$

$$\therefore AC = \frac{13}{\tan 34^\circ}$$

$$= 19{,}27\ldots$$

$$\text{In } \triangle BCD: \quad \frac{13}{BC} = \tan 28^\circ$$

$$\therefore BC = \frac{13}{\tan 28^\circ}$$

$$= 24{,}44\ldots$$

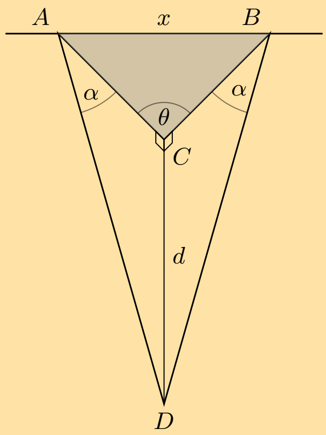
$$\text{Area } \triangle ABC = \frac{1}{2} AC \,.\, BC \sin \hat{C}$$

$$= \frac{1}{2}(19{,}27\ldots)(24{,}44\ldots)\sin 76^\circ$$

$$= 228{,}57\ldots$$

$$\approx 229 \text{ m}^2$$

b) If the platform is constructed so that the two angles of depression, $C\hat{A}D$ and $C\hat{B}D$, are both equal to 45° and the vertical depth of the gorge $CD = d$, $AB = x$ and $A\hat{C}B = \theta$, show that $\cos\theta = 1 - \frac{x^2}{2d^2}$.



Solution:

$$AB^2 = AC^2 + BC^2 - 2 \,.\, AC \,.\, BC \cos A\hat{C}B$$

$$\text{In } \triangle ACD: \quad \frac{d}{AC} = \tan 45^\circ$$

$$\therefore AC = d$$

$$\text{And } BC = d$$

$$\text{In } \triangle ABC: \quad x^2 = d^2 + d^2 - 2 \,.\, d \,.\, d \cos\theta$$

$$= 2d^2 - 2d^2 \cos\theta$$

$$2d^2 \cos\theta = 2d^2 - x^2$$

$$\therefore \cos\theta = \frac{2d^2 - x^2}{2d^2}$$

$$= 1 - \frac{x^2}{2d^2}$$

c) If $AB = 25$ m and $CD = 13$ m, calculate $A\hat{C}B$ (to the nearest integer).

Solution:

$$\begin{aligned}
\cos\theta &= 1 - \frac{x^2}{2d^2} \\
&= 1 - \frac{(25)^2}{2(13)^2} \\
\therefore \cos\theta &= -0{,}849\ldots \\
\text{ref } \angle &= 31{,}88^\circ \\
\therefore \theta &= 180^\circ - 31{,}88^\circ \\
&= 148{,}12^\circ \\
&\approx 148^\circ \qquad \text{(nearest integer)}
\end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28PP 2. 28PQ 3. 28PR 4. 28PS 5. 28PT

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4.6 Summary

Exercise 4 – 7: End of chapter exercises

1. Determine the following without using a calculator:

a) $\cos 15^\circ$

Solution:

$$\begin{aligned}
\cos 15^\circ &= \cos(60^\circ - 45^\circ) \\
&= \cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ \\
&= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) \\
&= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}} \\
&= \frac{1+\sqrt{3}}{2\sqrt{2}}
\end{aligned}$$

b) $\cos 75^\circ$

Solution:

$$\begin{aligned}
\cos 75^\circ &= \cos(45^\circ + 30^\circ) \\
&= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \\
&= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{3}-1}{2\sqrt{2}}
\end{aligned}$$

c) $\tan 75^\circ$

Solution:

$$\tan 75^\circ = \frac{\sin 75^\circ}{\cos 75^\circ}$$

$$\begin{aligned}\sin 75^\circ &= \sin(45^\circ + 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\\ \text{And from part b)}\quad \cos 75^\circ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ \therefore \frac{\sin 75^\circ}{\cos 75^\circ} &= \frac{\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{2\sqrt{2}}{\sqrt{3}-1}\\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1}\\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}\\ &= \frac{3+2\sqrt{3}+1}{3-1}\\ &= \frac{4+2\sqrt{3}}{2}\\ &= 2+\sqrt{3}\end{aligned}$$

d) $\cos 3^\circ \cos 42^\circ - \sin 3^\circ \sin 42^\circ$

Solution:

$$\begin{aligned}\cos 3^\circ \cos 42^\circ - \sin 3^\circ \sin 42^\circ &= \cos(3^\circ + 42^\circ)\\ &= \cos 45^\circ\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

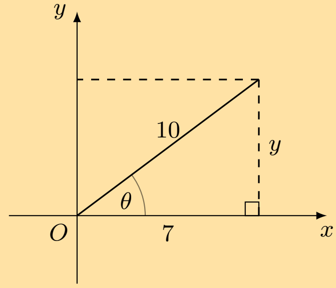
e) $1 - 2\sin^2(22{,}5^\circ)$

Solution:

$$\begin{aligned}1 - 2\sin^2(22{,}5^\circ) &= \cos 2(22{,}5^\circ)\\ &= \cos 45^\circ\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

2. Given $\cos\theta = 0{,}7$. Using a diagram, find $\cos 2\theta$ and $\cos 4\theta$.

Solution:



$$\cos\theta = 0{,}7 = \frac{7}{10}$$
$$\therefore y^2 = 10^2 - 7^2 = 51$$
$$\therefore y = \sqrt{51}$$
$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2(0{,}7)^2 - 1\\ &= -0{,}02\end{aligned}$$
$$\begin{aligned}\cos 4\theta &= \cos 2(2\theta)\\ &= 2\cos^2(2\theta) - 1\\ &= 2(-0{,}02)^2 - 1\\ &= -0{,}9992\end{aligned}$$

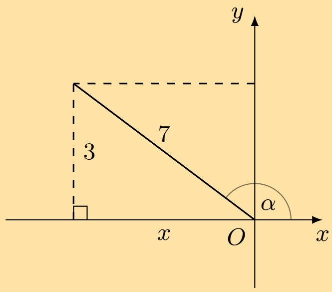
3. Given $7\sin\alpha = 3$ for $\alpha > 90^\circ$.

   Determine the following (leave answers in surd form):

   a) $\cos 2\alpha$

      Solution:

      Draw a sketch.



$$\sin\alpha = \frac{3}{7}$$
$$x^2 = 7^2 - 3^2 = 40$$
$$\therefore x = -\sqrt{40} \qquad (\alpha > 90^\circ)$$

$$\begin{aligned}\cos 2\alpha &= 2\cos^2\alpha - 1\\ &= 2\left(-\frac{\sqrt{40}}{7}\right)^2 - 1\\ &= 2\left(\frac{40}{49}\right) - 1\\ &= \frac{80}{49} - 1\\ &= \frac{31}{49}\end{aligned}$$

   b) $\tan 2\alpha$

      Solution:

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha}$$

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha \\ &= 2\left(\frac{3}{7}\right)\left(-\frac{\sqrt{40}}{7}\right) \\ &= -\frac{6\sqrt{40}}{49}\end{aligned}$$

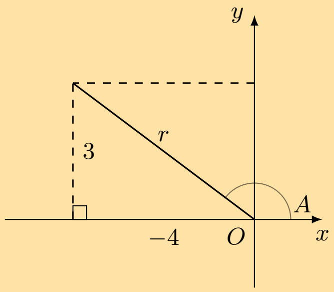
$$\begin{aligned}\tan 2\alpha &= \frac{\sin 2\alpha}{\cos 2\alpha} \\ &= \frac{-\frac{6\sqrt{40}}{49}}{\frac{31}{49}} \\ &= -\frac{6\sqrt{40}}{49} \times \frac{49}{31} \\ &= -\frac{6\sqrt{40}}{31}\end{aligned}$$

4. If $4\tan A + 3 = 0$ for $A < 270°$, determine, without the use of a calculator:

$$\left(\sin\frac{A}{2} - \cos\frac{A}{2}\right)\left(\sin\frac{A}{2} + \cos\frac{A}{2}\right)$$

Solution:

Draw a sketch.



$$\begin{aligned}4\tan A + 3 &= 0 \\ \tan A &= -\frac{3}{4}\end{aligned}$$

We are given that $A < 270°$, therefore A must lie in the second quadrant for the tangent function to be negative.

$$r^2 = 3^2 + (-4)^2 = 25$$
$$\therefore r = 5$$

$$\begin{aligned}
&\left(\sin\frac{A}{2} - \cos\frac{A}{2}\right)\left(\sin\frac{A}{2} + \cos\frac{A}{2}\right)\\
&= \sin^2\frac{A}{2} - \cos^2\frac{A}{2}\\
&= -\left(\cos^2\frac{A}{2} - \sin^2\frac{A}{2}\right)\\
&= -\cos 2\left(\frac{A}{2}\right)\\
&= -\cos A\\
&= -\left(-\frac{4}{5}\right)\\
&= \frac{4}{5}
\end{aligned}$$

5. Simplify: $\cos 67^\circ \cos 7^\circ + \cos 23^\circ \cos 83^\circ$

Solution:

$$\begin{aligned}
&\cos 67^\circ \cos 7^\circ + \cos 23^\circ \cos 83^\circ\\
&= \cos(90^\circ - 23^\circ)\cos(90^\circ - 83^\circ) + \cos 23^\circ \cos 83^\circ\\
&= \sin 23^\circ \sin 83^\circ + \cos 23^\circ \cos 83^\circ\\
&= \cos(83^\circ - 23^\circ)\\
&= \cos 60^\circ\\
&= \frac{1}{2}
\end{aligned}$$

6. Solve the equation:
$\cos 3\theta \cos\theta - \sin 3\theta \sin\theta = -\frac{1}{2}$ for $\theta \in [-90^\circ; 90^\circ]$.

Solution:

$$\begin{aligned}
\cos 3\theta \cos\theta - \sin 3\theta \sin\theta &= -\frac{1}{2}\\
\cos(3\theta + \theta) &= -\frac{1}{2}\\
\cos 4\theta &= -\frac{1}{2}\\
\text{ref } \angle &= 60^\circ\\
\text{Second quadrant:}\quad 4\theta &= 180^\circ - 60^\circ + k\ .\ 360^\circ\\
&= 120^\circ + k\ .\ 360^\circ\\
\therefore \theta &= 30^\circ + k\ .\ 90^\circ\\
\therefore \theta &= -60^\circ \text{ or } 30^\circ\\
\text{Third quadrant:}\quad 4\theta &= 180^\circ + 60^\circ + k\ .\ 360^\circ\\
&= 240^\circ + k\ .\ 360^\circ\\
\therefore \theta &= 60^\circ + k\ .\ 90^\circ\\
\therefore \theta &= -30^\circ \text{ or } 60^\circ\\
\text{Final answer:}\quad \theta &\in \{-60^\circ; -30^\circ; 30^\circ; 60^\circ\}
\end{aligned}$$

7. Find the general solution, without a calculator, for the following equations:

a) $3\sin\theta = 2\cos^2\theta$

Solution:

$$\begin{aligned}
3\sin\theta &= 2\cos^2\theta \\
3\sin\theta &= 2(1-\sin^2\theta) \\
3\sin\theta &= 2-2\sin^2\theta \\
2\sin^2\theta + 3\sin\theta - 2 &= 0 \\
\text{let } k &= \sin\theta \\
2k^2+3k-2 &= 0 \\
(k+2)(2k-1) &= 0 \\
\text{So } k = -2 \text{ or } k &= \frac{1}{2} \\
\text{if } k = -2,\ \sin\theta &= -2 \text{ which has no solution.} \\
\text{if } k = \frac{1}{2},\ \sin\theta &= \frac{1}{2} \\
\text{ref } \angle &= 30^\circ \\
\text{First quadrant: }\quad \theta &= 30^\circ + k\ .\ 360^\circ \\
\text{Second quadrant: }\quad \theta &= (180^\circ - 30^\circ) + k\ .\ 360^\circ \\
&= 150^\circ + k\ .\ 360^\circ, k \in \mathbb{Z} \\
\text{Final answer: } \theta = 30^\circ + k\ .\ 360^\circ &\text{ or } \theta = 150^\circ + k\ .\ 360^\circ, k \in \mathbb{Z}
\end{aligned}$$

b) $2\sin 2x - 2\cos x = \sqrt{2} - 2\sqrt{2}\sin x$

Solution:

$$\begin{aligned}
2\sin 2x - 2\cos x &= \sqrt{2} - 2\sqrt{2}\sin x \\
0 &= 2(2\sin x\cos x) - 2\cos x + 2\sqrt{2}\sin x - \sqrt{2} \\
0 &= 4\sin x\cos x - 2\cos x + 2\sqrt{2}\sin x - \sqrt{2} \\
0 &= 2\cos x(2\sin x - 1) + \sqrt{2}(2\sin x - 1) \\
0 &= (2\sin x - 1)(2\cos x + \sqrt{2}) \\
\text{If } 2\sin x - 1 &= 0 \\
\therefore \sin x &= \frac{1}{2} \\
\text{ref } \angle &= 30^\circ \\
\text{First quadrant: }\quad x &= 30^\circ + k\ .\ 360^\circ \\
\text{Second quadrant: }\quad x &= 150^\circ + k\ .\ 360^\circ, k \in \mathbb{Z}
\end{aligned}$$

$$\begin{aligned}
\text{If } 2\cos x + \sqrt{2} &= 0 \\
\therefore \cos x &= -\frac{\sqrt{2}}{2} \\
&= -\frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{\sqrt{2}} \\
&= -\frac{1}{\sqrt{2}} \\
\text{ref } \angle &= 45^\circ \\
\text{Second quadrant: }\quad x &= (180^\circ - 45^\circ) + k\ .\ 360^\circ \\
&= 135^\circ + k\ .\ 360^\circ \\
\text{Third quadrant: }\quad x &= (180^\circ + 45^\circ) + k\ .\ 360^\circ \\
&= 225^\circ + k\ .\ 360^\circ, k \in \mathbb{Z}
\end{aligned}$$

c) $\cos x\cos 10^\circ + \sin x\cos 100^\circ = 1 - 2\sin^2 x$

Solution:

$$\cos x \cos 10^\circ + \sin x \cos 100^\circ = 1 - 2\sin^2 x$$
$$\cos x \cos 10^\circ + \sin x \cos(90^\circ + 10^\circ) = \cos 2x$$
$$\cos x \cos 10^\circ - \sin x \sin 10^\circ = \cos 2x$$
$$\cos(x + 10^\circ) = \cos 2x$$

First quadrant: $x + 10^\circ = 2x + k \ . \ 360^\circ$

$x = 10^\circ + k \ . \ 360^\circ$

Fourth quadrant: $x + 10^\circ = (360^\circ - 2x) + k \ . \ 360^\circ$

$3x = 350^\circ + k \ . \ 360^\circ$

$\therefore x = 116{,}7^\circ + k \ . \ 120^\circ$

Final answer: $x = 10^\circ + k \ . \ 360^\circ$

$x = 116{,}7^\circ + k \ . \ 120^\circ, k \in \mathbb{Z}$

d) $6\sin^2 \alpha + 2\sin 2\alpha - 1 = 0$

Solution:

$$6\sin^2 \alpha + 2\sin 2\alpha - 1 = 0$$
$$6\sin^2 \alpha + 2(2\sin \alpha \cos \alpha) - 1 = 0$$
$$6\sin^2 \alpha + 4\sin \alpha \cos \alpha - (\sin^2 \alpha + \cos^2 \alpha) = 0$$
$$6\sin^2 \alpha + 4\sin \alpha \cos \alpha - \sin^2 \alpha - \cos^2 \alpha = 0$$
$$5\sin^2 \alpha + 4\sin \alpha \cos \alpha - \cos^2 \alpha = 0$$
$$(5\sin \alpha - \cos \alpha)(\sin \alpha + \cos \alpha) = 0$$

If $5\sin \alpha - \cos \alpha = 0$

$5\sin \alpha = \cos \alpha$

$\therefore \tan \alpha = \frac{1}{5}$

$\therefore \alpha = 11{,}3^\circ + k \ . \ 180^\circ$

If $\sin \alpha + \cos \alpha = 0$

$\sin \alpha = -\cos \alpha$

$\therefore \tan \alpha = -1$

ref $\angle = 45^\circ$

Second quadrant: $\alpha = (180^\circ - 45^\circ) + k \ . \ 180^\circ$

$= 135^\circ + k \ . \ 180^\circ, k \in \mathbb{Z}$

Final answer: $\alpha = 11{,}3^\circ + k \ . \ 180^\circ$

$\alpha = 135^\circ + k \ . \ 180^\circ, k \in \mathbb{Z}$

8. a) Prove: $\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$

Solution:

$$
\begin{aligned}
\text{RHS} &= \frac{1}{4}\left(3\sin\theta - \sin[2\theta+\theta]\right) \\
&= \frac{1}{4}(3\sin\theta - (\sin 2\theta\cos\theta + \cos 2\theta\sin\theta)) \\
&= \frac{1}{4}(3\sin\theta - \sin 2\theta\cos\theta - \cos 2\theta\sin\theta) \\
&= \frac{1}{4}(3\sin\theta - 2\sin\theta\cos\theta\cos\theta - \sin\theta(1-2\sin^2\theta)) \\
&= \frac{1}{4}(3\sin\theta - 2\sin\theta\cos^2\theta - \sin\theta + 2\sin^3\theta) \\
&= \frac{1}{4}(2\sin\theta - 2\sin\theta\cos^2\theta + 2\sin^3\theta) \\
&= \frac{1}{4}(2\sin\theta - 2\sin\theta(1-\sin^2\theta) + 2\sin^3\theta) \\
&= \frac{1}{4}(2\sin\theta - 2\sin\theta + 2\sin^3\theta + 2\sin^3\theta) \\
&= \frac{1}{4}(4\sin^3\theta) \\
&= \sin^3\theta \\
&= \text{LHS}
\end{aligned}
$$

b) Hence, solve the equation $3\sin\theta - \sin 3\theta = 2$ for $\theta \in [0°; 360°]$.

Solution:

$$
\begin{aligned}
3\sin\theta - \sin 3\theta &= 2 \\
\frac{3\sin\theta - \sin 3\theta}{4} &= \frac{2}{4} \\
\frac{3\sin\theta - \sin 3\theta}{4} &= \frac{1}{2} \\
\therefore \sin^3\theta &= \frac{1}{2} \\
\therefore \sin\theta &= 0{,}793\ldots \\
\text{ref}\,\angle &= 52{,}53° \\
\text{First quadrant: } \theta &= 52{,}53° + k\,.\,360°, k \in \mathbb{Z}
\end{aligned}
$$

$$
\begin{aligned}
\text{Second quadrant: } \theta &= (180° - 52{,}53°) + k\,.\,180° \\
&= 127{,}47° + k\,.\,360°, k \in \mathbb{Z}
\end{aligned}
$$

$$
\begin{aligned}
\text{Final answer: } \theta &= 52{,}53° \\
\theta &= 127{,}47°
\end{aligned}
$$

9. Prove the following identities:

a) $\cos^2\alpha\left(1-\tan^2\alpha\right) = \cos 2\alpha$

Solution:

$$
\begin{aligned}
\text{LHS} &= \cos^2\alpha(1-\tan^2\alpha) \\
&= \cos^2\alpha - \cos^2\alpha\left(\frac{\sin^2\alpha}{\cos^2\alpha}\right) \\
&= \cos^2\alpha - \sin^2\alpha \\
&= \cos 2\alpha \\
&= \text{RHS}
\end{aligned}
$$

Restrictions: $\alpha \neq 90° + k\,.\,180°, \quad k \in \mathbb{Z}$.

b) $4\sin\theta\cos\theta\cos 2\theta = \sin 4\theta$

Solution:

$$\begin{aligned}
\text{RHS} &= \sin 4\theta \\
&= \sin 2(2\theta) \\
&= 2\sin 2\theta\cos 2\theta \\
&= 2(2\sin\theta\cos\theta)\cos 2\theta \\
&= 4\sin\theta\cos\theta\cos 2\theta \\
&= \text{ LHS}
\end{aligned}$$

c) $4\cos^3 x - 3\cos x = \cos 3x$

Solution:

$$\begin{aligned}
\text{RHS} &= \cos 3x \\
&= \cos(2x + x) \\
&= \cos 2x\cos x - \sin 2x\sin x \\
&= \cos 2x\cos x - 2\sin^2 x\cos x \\
&= \cos x(\cos 2x - 2\sin^2 x) \\
&= \cos x(2\cos^2 x - 1 - 2[1 - \cos^2 x]) \\
&= \cos x(2\cos^2 x - 1 - 2 + 2\cos^2 x) \\
&= \cos x(4\cos^2 x - 3) \\
&= 4\cos^3 x - 3\cos x \\
&= \text{ LHS}
\end{aligned}$$

d) $\cos 2A + 2\sin 2A + 2 = (3\cos A + \sin A)(\cos A + \sin A)$

Solution:

$$\begin{aligned}
\text{LHS} &= \cos 2A + 2\sin 2A + 2 \\
&= (\cos^2 A - \sin^2 A) + (4\sin A\cos A) + 2(\cos^2 A + \sin^2 A) \\
&= \cos^2 A - \sin^2 A + 4\sin A\cos A + 2\cos^2 A + 2\sin^2 A) \\
&= 3\cos^2 A + 4\sin A\cos A + \sin^2 A \\
&= (3\cos A + \sin A)(\cos A + \sin A) \\
&= \text{ RHS}
\end{aligned}$$

e) $\dfrac{\cos 2x}{(\cos x + \sin x)^3} = \dfrac{\cos x - \sin x}{1 + \sin 2x}$

Solution:

$$\begin{aligned}
\text{LHS} &= \frac{\cos 2x}{(\cos x + \sin x)^3} \\
&= \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^3} \\
&= \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\cos x + \sin x)^2(\cos x + \sin x)} \\
&= \frac{\cos x - \sin x}{(\cos x + \sin x)^2} \\
&= \frac{\cos x - \sin x}{\cos^2 x + 2\cos x\sin x + \sin^2 x} \\
&= \frac{\cos x - \sin x}{1 + 2\cos x\sin x} \\
&= \frac{\cos x - \sin x}{1 + \sin 2x} \\
&= \text{ RHS}
\end{aligned}$$

10. a) Prove: $\tan y = \frac{\sin 2y}{\cos 2y + 1}$

Solution:

$$\begin{aligned} \text{RHS} &= \frac{\sin 2y}{\cos 2y + 1} \\ &= \frac{\sin 2y}{(2\cos^2 y - 1) + 1} \\ &= \frac{2\sin y \cos y}{2\cos^2 y} \\ &= \frac{\sin y}{\cos y} \\ &= \tan y \\ &= \text{LHS} \end{aligned}$$

b) For which values of y is the identity undefined?

Solution:

The identity is undefined for the values of y such that $\cos 2y + 1 = 0$ since division by zero is not permitted.

$$\begin{aligned} \text{If } \cos 2y + 1 &= 0 \\ \cos 2y &= -1 \\ \therefore 2y &= 180^\circ + k \;.\; 360^\circ \\ \therefore y &= 90^\circ + k \;.\; 180^\circ, k \in \mathbb{Z} \end{aligned}$$

Restricted values are: $y = 90^\circ + k \;.\; 180^\circ, k \in \mathbb{Z}$

11. Given: $1 + \tan^2 3\theta - 3\tan 3\theta = 5$

a) Find the general solution.

Solution:

$$\begin{aligned} 1 + \tan^2 3\theta - 3\tan 3\theta &= 5 \\ \tan^2 3\theta - 3\tan 3\theta - 4 &= 0 \\ (\tan 3\theta - 4)(\tan 3\theta + 1) &= 0 \\ \text{If } \tan 3\theta &= -1 \\ \text{ref } \angle &= 45^\circ \\ \therefore 3\theta &= (180^\circ - 45^\circ) + k \;.\; 180^\circ \\ 3\theta &= 135^\circ + k \;.\; 180^\circ \\ \therefore \theta &= 45^\circ + k \;.\; 60^\circ, k \in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} \text{If } \tan 3\theta &= 4 \\ \text{ref } \angle &= 75{,}96^\circ \\ \therefore 3\theta &= 75{,}96^\circ + k \;.\; 180^\circ \\ \therefore \theta &= 25{,}32^\circ + k \;.\; 60^\circ, k \in \mathbb{Z} \end{aligned}$$

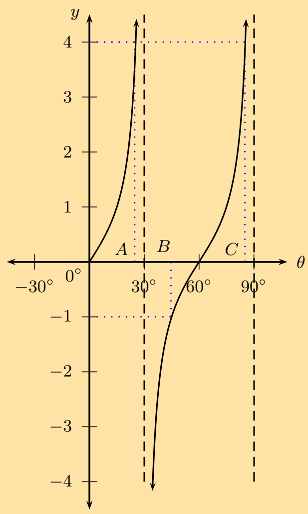
b) Find the solution for $\theta \in [0^\circ; 90^\circ]$.

Solution:

$$\begin{aligned} k = 0 : \quad \theta\theta &= 45^\circ \\ &= 25{,}32^\circ \\ k = 1 : \quad \theta &= 85{,}32^\circ \end{aligned}$$

c) Draw a graph of $y = \tan 3\theta$ for $\theta \in [0^\circ; 90^\circ]$ and indicate the solutions to the equation on the graph.

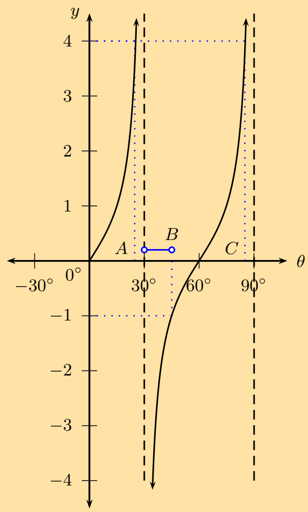
Solution:



$$\text{Period: } = 60^\circ$$
$$\text{Asymptote: } \theta = 30^\circ$$
$$\text{Asymptote: } \theta = 90^\circ$$
$$A(25{,}32^\circ; 4)$$
$$B(45^\circ; -1)$$
$$C(85{,}32^\circ; 4)$$

d) Use the graph to determine where $\tan 3\theta < -1$.

Solution:



$30^\circ < \theta < 45^\circ$

12. a) Show that:

$$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$$

Solution:

$$\begin{aligned}
\text{LHS} &= \sin(A+B) - \sin(A-B) \\
&= \sin A\cos B + \cos A\sin B - [\sin A\cos B - \cos A\sin B] \\
&= \sin A\cos B + \cos A\sin B - \sin A\cos B + \cos A\sin B \\
&= 2\cos A\sin B \\
&= \text{RHS}
\end{aligned}$$

b) Use this result to solve $\sin 3x - \sin x = 0$ for $x \in [-180°; 360°]$.

Solution:

$$\begin{aligned}
\sin 3x - \sin x &= 0 \\
\sin(2x + x) - \sin(2x - x) &= 0 \\
\text{So } A = 2x &\text{ and } B = x \\
\therefore \text{ we can write } 2\cos 2x \sin x &= 0
\end{aligned}$$

$$\begin{aligned}
\text{If } \cos 2x &= 0 \\
2x &= 90° + k\ .\ 360° \\
\therefore x &= 45° + k\ .\ 180°, k \in \mathbb{Z}
\end{aligned}$$

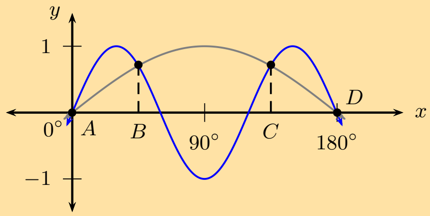
$$\begin{aligned}
2x &= 270° + k\ .\ 360° \\
\therefore x &= 135° + k\ .\ 180°, k \in \mathbb{Z}
\end{aligned}$$

$$\begin{aligned}
\text{If } \sin x &= 0 \\
x &= 0° + k\ .\ 360°, k \in \mathbb{Z} \\
\text{or } x &= 180° + k\ .\ 180°, k \in \mathbb{Z}
\end{aligned}$$

$$\begin{aligned}
\text{Final answer: } k = -2: &\quad x = -180° \\
k = -1: &\quad x = -135°; -45° \\
k = 0: &\quad x = 45°; 135°; 0°; 180° \\
k = 1: &\quad x = 225°; 315°; 360° \\
\text{Final answer: } &x = -180°; -135°; -45°; \\
&0°; 45°; 135°; 180°; \\
&225°; 315°; 360°
\end{aligned}$$

c) On the same system of axes, draw two graphs to solve graphically: $\sin 3x - \sin x = 0$ for $x \in [0°; 360°]$. Indicate the solutions on the graph using the letters $A, B, \ldots$ etc.

Solution:



13. Given: $\cos 2x = \sin x$ for $x \in [0°; 360°]$

a) Solve for x algebraically.

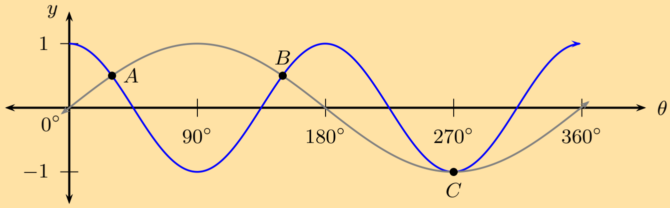
Solution:

$$\cos 2x = \sin x$$
$$\cos 2x = \cos(90^\circ - x)$$
$$2x = 90^\circ - x + k \,.\, 360^\circ$$
$$3x = 90^\circ + k \,.\, 360^\circ$$
$$x = 30^\circ + k \,.\, 120^\circ$$

$$2x = [360^\circ - (90^\circ - x)] + k \,.\, 360^\circ$$
$$= 270^\circ + x + k \,.\, 360^\circ$$
$$x = 270^\circ + k \,.\, 360^\circ$$

$k = 0:$ $x = 30^\circ$
$x = 270^\circ$
$k = 1:$ $x = 150^\circ$
$x = 270^\circ$

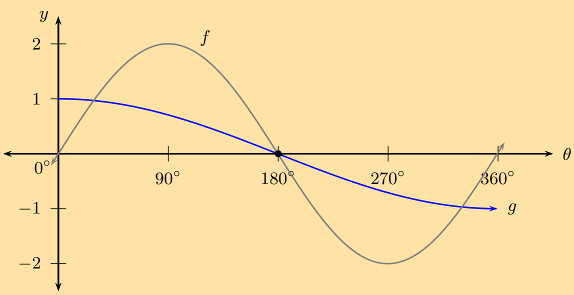
b) Verify the solution graphically by sketching two graphs on the same system of axes.
Solution:



$y = \cos 2x$ (blue graph)
$y = \sin x$ (grey graph)
$A\left(30^\circ; \frac{1}{2}\right)$
$B\left(150^\circ; \frac{1}{2}\right)$
$C\,(270^\circ; -1)$

14. The following graphs are given below:

$$f := a \sin x$$
$$g := \cos bx \qquad (x \in [0^\circ; 360^\circ])$$



a) Explain why $a = 2$ and $b = \frac{1}{2}$.
Solution:
From the graph we can see that the amplitude of the sine graph is 2, therefore $a = 2$.
For the cosine graph, the period is 720°, therefore $b = \frac{360^\circ}{720^\circ} = \frac{1}{2}$.

b) For how many x-values in $[0^\circ; 360^\circ]$ will $f(x) - g(x) = 0$?

Solution:
For:

$$f(x) - g(x) = 0$$
$$f(x) = g(x)$$

For the interval $[0°; 360°]$, the diagram shows that the two graphs intersect at three places.

c) Use the graph to solve $f(x) - g(x) = 1$.
Solution:
From the graph we have the following solution:

$$f(360°) - g(360°) = 0 - (-1) = 1$$
$$\therefore x = 360°$$

d) Solve $a \sin x = \cos bx$ for $x \in [0°; 360°]$ using trigonometric identities.
Solution:

$$2 \sin x = \cos \frac{1}{2}x$$
$$2 \sin 2\left(\frac{x}{2}\right) = \cos \frac{1}{2}x$$
$$2\left(2 \sin \frac{x}{2} \cos \frac{x}{2}\right) = \cos \frac{1}{2}x$$
$$4 \sin \frac{x}{2} \cos \frac{x}{2} - \cos \frac{1}{2}x = 0$$
$$\cos \frac{x}{2}\left(4 \sin \frac{x}{2} - 1\right) = 0$$

$$\text{If } \cos \frac{x}{2} = 0$$
$$\frac{x}{2} = 90°$$
$$\therefore x = 180°$$

$$\text{If } 4 \sin \frac{x}{2} - 1 = 0$$
$$4 \sin \frac{x}{2} = 1$$
$$\sin \frac{x}{2} = \frac{1}{4}$$
$$\text{ref } \angle = 14{,}47°$$
$$\therefore \frac{x}{2} = 14{,}47°$$
$$x = 29°$$
$$\text{or } \frac{x}{2} = 180° - 14{,}47°$$
$$= 165{,}53°$$
$$\therefore x = 331°$$

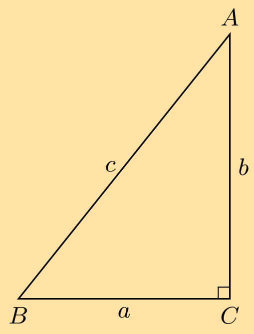
Final answer: $x = 180°, 29°, 331°$

e) For which values of x will $\frac{1}{2} \cos\left(\frac{x}{2}\right) \leq \sin x$ for $x \in [0°; 360°]$?
Solution:

$$\frac{1}{2} \cos\left(\frac{x}{2}\right) \leq \sin x$$
$$\cos\left(\frac{x}{2}\right) \leq 2 \sin x$$
$$\text{where } g(x) \leq f(x)$$

Answer: $29° \leq x \leq 180°$ or $331° \leq x \leq 360°$

15. In $\triangle ABC$, $AB = c, BC = a, CA = b$ and $\hat{C} = 90°$



a) Prove that $\sin 2A = \frac{2ab}{c^2}$.

Solution:

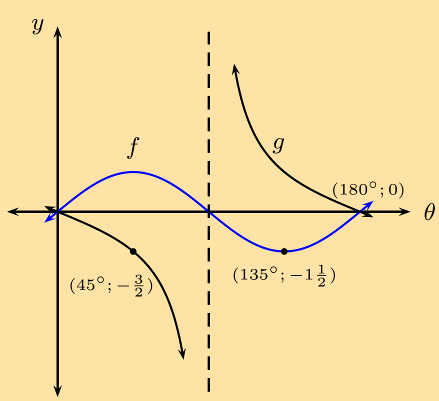
$$\begin{aligned} \text{LHS} &= \sin 2A \\ &= 2 \sin A \cos A \\ &= 2\left(\frac{a}{c}\right)\left(\frac{b}{c}\right) \\ &= \frac{2ab}{c^2} \\ &= \text{RHS} \end{aligned}$$

b) Show that $\cos 2A = \frac{b^2 - a^2}{c^2}$.

Solution:

$$\begin{aligned} \text{LHS} &= \cos 2A \\ &= \cos^2 A - \sin^2 A \\ &= \left(\frac{b}{c}\right)^2 - \left(\frac{a}{c}\right)^2 \\ &= \frac{b^2 - a^2}{c^2} \\ &= \text{RHS} \end{aligned}$$

16. Given the graphs of $f(\theta) = p \sin k\theta$ and $g(\theta) = q \tan \theta$, determine the values of p, k and q.

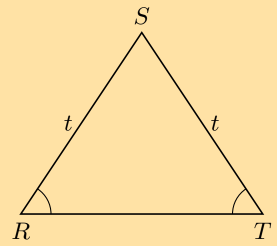


Solution:

$f(\theta) = \frac{3}{2} \sin 2\theta$ and $g(\theta) = -\frac{3}{2} \tan \theta$

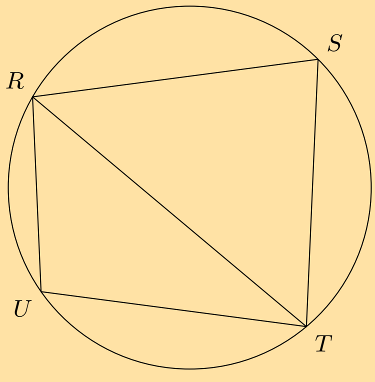
17. $\triangle RST$ is an acute angled triangle with $RS = ST = t$. Show that area $\triangle RST = t^2 \sin \hat{T} \cos \hat{T}$.

Solution:



$$\begin{aligned}
\text{Area } \triangle RST &= \frac{1}{2}t \cdot t \cdot \sin \hat{S} \\
&= \frac{1}{2}t^2 \sin[180^\circ - \left(\hat{R} + \hat{T}\right)] \\
&= \frac{1}{2}t^2 \sin \left(\hat{R} + \hat{T}\right) \\
\text{and } \hat{R} &= \hat{T} \qquad (\angle\text{s opp. equal sides}) \\
\therefore \text{Area } \triangle RST &= \frac{1}{2}t^2 \sin \left(2\hat{T}\right) \\
&= \frac{1}{2}t^2 (2 \sin \hat{T} \cos \hat{T}) \\
&= t^2 \sin \hat{T} \cos \hat{T}
\end{aligned}$$

18. $RSTU$ is a cyclic quadrilateral with $RU = 6$ cm, $UT = 7{,}5$ cm, $RT = 11$ cm and $RS = 9{,}5$ cm.



a) Calculate $\hat{U}$.

Solution:

$$\begin{aligned}
\text{In } \triangle RUT \quad RT^2 &= RU^2 + UT^2 - 2RU \cdot UT \cos \hat{U} \\
\therefore \cos \hat{U} &= \frac{RU^2 + UT^2 - RT^2}{2RU \cdot UT} \\
&= \frac{6^2 + (7{,}5)^2 - 11^2}{2(6)(7{,}5)} \\
&= -0{,}3194\ldots \\
\text{ref } \angle &= 71{,}4^\circ \\
\hat{U} &= 180^\circ - 71{,}4^\circ \\
&= 108{,}6^\circ
\end{aligned}$$

b) Determine $\hat{S}$.

Solution:

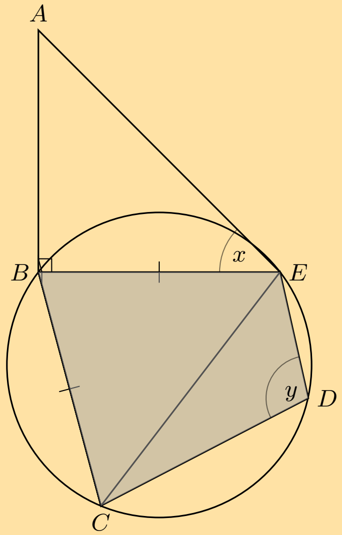
$$\begin{aligned}\hat{S} &= 180° - 108{,}6° \quad \text{(opp. } \angle\text{s of cyclic. quad. suppl.)}\\ &= 71{,}4°\end{aligned}$$

c) Find $R\hat{T}S$.

Solution:

$$\begin{aligned}\frac{\sin R\hat{T}S}{RS} &= \frac{\sin R\hat{S}T}{RT}\\ \frac{\sin R\hat{T}S}{9{,}5} &= \frac{\sin 71{,}4}{11}\\ \sin R\hat{T}S &= \frac{9{,}5 \sin 71{,}4}{11}\\ &= 0{,}8185\ldots\\ \therefore R\hat{T}S &= 54{,}9°\end{aligned}$$

19. $BCDE$ is a cyclic quadrilateral that lies in a horizontal plane. AB is a vertical pole with base B. The angle of elevation from E to A is $x°$ and $C\hat{D}E = y°$. $\triangle BEC$ is an isosceles triangle with $BE = BC$.



a) Show that $B\hat{C}E = \frac{1}{2}y$.

Solution:

In cyclic quadrilateral $BCDE$:

$$\begin{aligned}\hat{B} &= 180° - y\\ BE &= BC \quad \text{(given)}\\ B\hat{C}E &= B\hat{E}C\\ &= \frac{180° - (180° - y)}{2}\\ &= \frac{y}{2}\end{aligned}$$

b) Show that $CE = 2BE\cos\left(\frac{y}{2}\right)$

Solution:

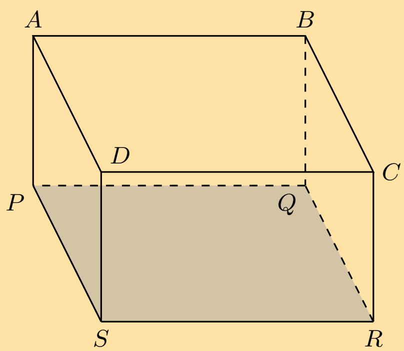
$$\text{In } \triangle BCE: \quad \frac{CE}{\sin C\hat{B}E} = \frac{BE}{\sin B\hat{C}E}$$
$$\therefore CE = \frac{BE \sin C\hat{B}E}{\sin B\hat{C}E}$$
$$= \frac{BE \sin(180° - y)}{\sin\left(\frac{y}{2}\right)}$$
$$= \frac{BE \sin y}{\sin\left(\frac{y}{2}\right)}$$
$$= \frac{BE(2\sin\left(\frac{y}{2}\right)\cos\left(\frac{y}{2}\right))}{\sin\left(\frac{y}{2}\right)}$$
$$= 2BE\cos\left(\frac{y}{2}\right)$$

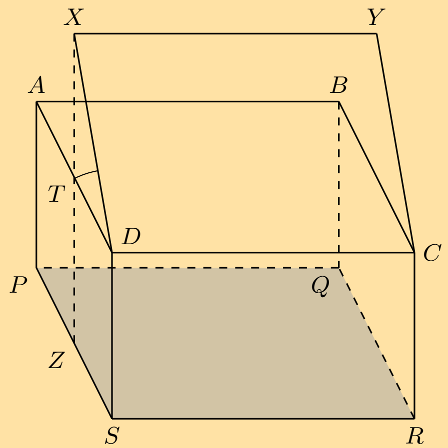
c) If $AB = 2{,}6$ m, $x = 37°$ and $y = 109°$, calculate the length of CE.

Solution:

$$\text{In } \triangle ABE: \quad \frac{AB}{BE} = \tan x \qquad (A\hat{B}E = 90°)$$
$$\therefore BE = \frac{AB}{\tan x}$$
$$= \frac{2{,}6}{\tan 37°}$$
$$= 3{,}45$$
$$\therefore CE = 2BE\cos\left(\frac{y}{2}\right)$$
$$= 2(3{,}45)\cos\left(\frac{109°}{2}\right)$$
$$= 4 \text{ m}$$

20. The first diagram shows a rectangular box with $SR = 8$ cm, $PS = 6$ cm and $PA = 4$ cm. The lid of the box, $ABCD$, is opened 30° to the position $XYCD$, as shown in the second diagram.





a) Write down the dimensions (length, breadth and diagonal) of the lid $XYCD$.

Solution:

$$\begin{aligned} \text{length } &= XY = DC = 8 \text{ cm} \\ \text{breadth } &= XD = YC = 6 \text{ cm} \\ \text{diagonal } &= AC = XC \\ &= \sqrt{8^2 + 6^2} \qquad \text{(Pythagoras)} \\ &= 10 \text{ cm} \end{aligned}$$

b) Calculate XZ, the perpendicular height of X above the base of the box.

Solution:

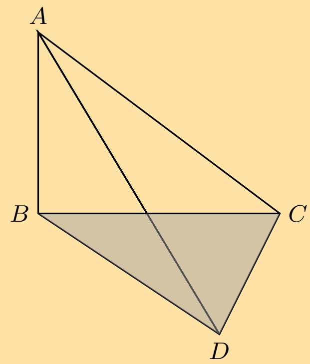
$$\begin{aligned} TZ &= AP = 4 \text{ cm} \\ \text{In } \triangle XTD: \quad \frac{XT}{XD} &= \sin 30^\circ \\ \frac{XT}{6} &= \frac{1}{2} \\ \therefore XT &= 3 \text{ cm} \\ \therefore XZ &= 4 + 3 = 7 \text{ cm} \end{aligned}$$

c) Calculate the ratio $\frac{\sin X\hat{Z}C}{\sin X\hat{C}Z}$.

Solution:

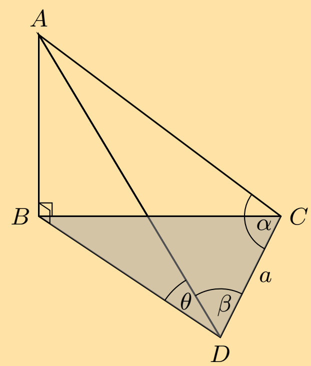
$$\begin{aligned} \text{In } \triangle XTC: \quad \frac{\sin X\hat{Z}C}{XC} &= \frac{\sin X\hat{C}Z}{XZ} \\ \therefore \frac{\sin X\hat{Z}C}{\sin X\hat{C}Z} &= \frac{XC}{XZ} \\ &= \frac{10}{7} \end{aligned}$$

21. AB is a vertical pole on a horizontal plane BCD. DC is a metres and the angle of elevation from D to A is θ. $A\hat{C}D = \alpha$ and $A\hat{D}C = \beta$.



a) Name the two right angles in the diagram.

Solution:



$$A\hat{B}C = 90^\circ$$
$$A\hat{B}D = 90^\circ$$

b) Show that $AB = \frac{a \sin\alpha \sin\theta}{\sin(\alpha+\beta)}$.

Solution:

$$\text{In } \triangle ABC: \quad \frac{AB}{AD} = \sin\theta$$
$$\therefore AB = AD \sin\theta$$
$$\text{In } \triangle ADC: \quad \frac{AD}{\sin\alpha} = \frac{DC}{\sin D\hat{A}C}$$
$$AD = \frac{a \sin\alpha}{\sin[180^\circ - (\alpha+\beta)]}$$
$$= \frac{a\sin\alpha}{\sin(\alpha+\beta)}$$

$$\therefore AB = \frac{a\sin\alpha\sin\theta}{\sin(\alpha+\beta)}$$

c) If it is given that $AD = AC$, show that the height of the pole is given by $AB = \frac{a\sin\theta}{2\cos\alpha}$.

Solution:

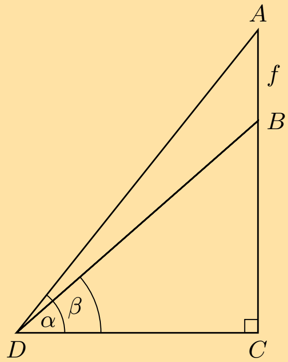
$$\begin{aligned} AD &= AC \qquad \text{(given)} \\ \therefore \beta &= \alpha \qquad \text{(isos. } \triangle ADC) \\ \therefore AB &= \frac{a \sin\alpha \sin\theta}{\sin(\alpha+\alpha)} \\ &= \frac{a \sin\alpha \sin\theta}{\sin 2\alpha} \\ &= \frac{a \sin\alpha \sin\theta}{2 \sin\alpha \cos\alpha} \\ &= \frac{a \sin\theta}{2\cos\alpha} \end{aligned}$$

d) Calculate the height of the pole if $a = 13$ m, $\theta = 33°, \alpha = \beta = 65°$.

Solution:

$$\begin{aligned} AB &= \frac{a \sin\theta}{2\cos\alpha} \\ &= \frac{13 \times \sin 33°}{2\cos 65°} \\ &= 8{,}4 \text{ m} \end{aligned}$$

22. AB is a flagpole on top of a government building BC. $AB = f$ units and D is a point on the ground in the same horizontal plane as the base of the building, C. The angle of elevation from D to A and B is α and β, respectively.



a) Show that $f = \frac{BC \sin(\alpha-\beta)}{\sin\beta \cos\alpha}$

Solution:

$$\text{In } \triangle ABD: \quad \frac{f}{\sin(\alpha-\beta)} = \frac{DB}{\sin(90° - \alpha)}$$

$$\therefore f = \frac{DB \sin(\alpha-\beta)}{\cos\alpha}$$

$$\text{In } \triangle BDC: \quad \frac{BC}{DB} = \sin\beta$$

$$\therefore DB = \frac{BC}{\sin\beta}$$

$$\therefore f = \frac{BC \sin(\alpha-\beta)}{\sin\beta\cos\alpha}$$

b) Calculate the height of the flagpole (to the nearest metre) if the building is 7 m, $\alpha = 63°$ and $\beta = 57°$.

Solution:

$$\begin{aligned} f &= \frac{BC \sin(\alpha - \beta)}{\sin \beta \cos \alpha} \\ &= \frac{7 \times \sin(63° - 57°)}{\sin 57° \cos 63°} \\ &= 1{,}92 \end{aligned}$$

$\therefore$ height $\approx$ 2 m (nearest metre)

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28PV	1b. 28PW	1c. 28PX	1d. 28PY	1e. 28PZ	2. 28Q2
3a. 28Q3	3b. 28Q4	4. 28Q5	5. 28Q6	6. 28Q7	7a. 28Q8
7b. 28Q9	7c. 28QB	7d. 28QC	8. 28QD	9a. 28QF	9b. 28QG
9c. 28QH	9d. 28QJ	9e. 28QK	10. 28QM	11. 28QN	12. 28QP
13. 28QQ	14. 28QR	15. 28QS	16. 28QT	17. 28QV	18. 28QW
19. 28QX	20. 28QY	21. 28QZ	22. 28R2		

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CHAPTER 5

Polynomials

- Proofs of theorems are not examinable.
- Be careful of spending too much time on revision. Learners should be familiar with these factorisation methods from Grade 10 and 11.
- Explain terminology carefully. It is very important for learners to understand what a factor is and how it relates to the graphs of quadratic and cubic functions (covered in Differential Calculus chapter).
- Long division and synthetic division are included as an introduction to the remainder theorem. Do not spend too much time on these techniques.
- It is important for learners to understand that a remainder of zero means that divisor is a factor.
- Encourage learners to practise factorising by inspection.

5.1 Revision

Identifying polynomials

Exercise 5 – 1: Identifying polynomials

1. Given $f(x) = 2x^3 + 3x^2 - 1$, determine whether the following statements are true or false. If false, provide the correct statement.

   a) $f(x)$ is a trinomial.

   Solution: True

   b) The coefficient of the x is zero.

   Solution: True

   c) $f\left(\frac{1}{2}\right) = \frac{1}{12}$.

   Solution:

$$\begin{aligned} f(x) &= 2x^3 + 3x^2 - 1 \\ f\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 - 1 \\ &= 2\left(\frac{1}{8}\right) + 3\left(\frac{1}{4}\right) - 1 \\ &= \frac{1}{4} + \frac{3}{4} - 1 \\ &= 1 - 1 \\ &= 0 \\ \therefore & \text{ False} \end{aligned}$$

   d) $f(x)$ is of degree 3.

   Solution: True

   e) The constant term is 1.

   Solution: False: -1

   f) $f(x)$ will have 3 real roots.

   Solution: True

2. Given $g(x) = 2x^3 - 9x^2 + 7x + 6$, determine the following:

   a) the number of terms in $g(x)$.

Solution: 4

b) the degree of $g(x)$.

Solution: 3

c) the coefficient of the x^2 term.

Solution: -9

d) the constant term.

Solution: 6

3. Determine which of the following expressions are polynomials and which are not. For those that are not polynomials, give reasons.

a) $y^3 + \sqrt{5}$

Solution: Cubic polynomial

b) $-x^2 - x - 1$

Solution: Quadratic polynomial

c) $4\sqrt{k} - 9$

Solution: Not a polynomial; in $k^{\frac{1}{2}}$ the exponent is not a natural number.

d) $\frac{2}{p} + p + 3$

Solution: Not a polynomial; in p^{-1} the exponent is not a natural number.

e) $x(x-1)(x-2) - 2$

Solution: Cubic polynomial

f) $\left(\sqrt{m} - 1\right)\left(\sqrt{m} + 1\right)$

Solution:

$$\begin{aligned}\left(\sqrt{m} - 1\right)\left(\sqrt{m} + 1\right) &= \left(\sqrt{m}\right)^2 + \sqrt{m} - \sqrt{m} - 1 \\ &= m - 1\end{aligned}$$

Linear polynomial

g) $t^0 - 1$

Solution:

$$\begin{aligned}t^0 - 1 &= 1 - 1 \\ &= 0\end{aligned}$$

Zero polynomial

h) $16y^7$

Solution: Polynomial; degree 7

i) $-\frac{x^3}{2} + 5x^2 + \frac{x}{3} - 11$

Solution: Cubic polynomial

j) $4b^0 + 3b^{-1} + 5b^2 - b^3$

Solution: Not a polynomial; in b^{-1} the exponent is not a natural number.

4. Peter's mathematics homework is shown below. Find and correct his mistakes.

Homework:

Given $p(x) = x + \dfrac{4}{x} - 5$, answer the following questions:

a) Simplify the expression.

b) Is $p(x)$ a polynomial?

c) What is the coefficient of the x term?

Peter's answers:

a)

$$\begin{aligned}p(x) &= x + \tfrac{4}{x} - 5 && \text{(restrictions: } x \neq 0) \\ &= x^2 + 4 - 5x && \text{(multiply through by } x) \\ &= x^2 - 5x + 4 && \text{(write in descending order)} \\ &= (x-1)(x+4) && \text{(factorise, quadratic has two roots)}\end{aligned}$$

b) Yes, because it can be simplified to have exponents that are all natural numbers. It is a quadratic binomial because the highest exponent is two and there are only two terms; $(x - 1)$ and $(x + 4)$.

c) Before I simplified, the coefficient of the x term was nothing and after I simplified it became 5.

Solution:

Peter made the following mistakes:

a) This is not an equation, therefore Peter must not multiply through by x. Peter did not factorise correctly, he made an error with the sign in the second bracket .

$$\begin{aligned} p(x) &= x + \tfrac{4}{x} - 5 && \text{(restrictions: } x \neq 0) \\ &= \tfrac{x^2+4-5x}{x} && \text{(common denominator is } x) \\ &= \tfrac{x^2-5x+4}{x} && \text{(write in standard form)} \\ &= \tfrac{(x-1)(x-4)}{x} && \text{(factorise and be careful of the signs)} \end{aligned}$$

b) The original expression is not a polynomial because the exponent of the second term $\left(\frac{4}{x} = 4x^{-1}\right)$ is not a natural number. Peter's simplified expression is a quadratic trinomial because it has three terms and two factors.

c) In the original expression, the coefficient of the x term is 1. After Peter wrongly simplified the expression, the coefficient of the x term was -5 and not $+5$. Always remember that the coefficient includes the sign and the number in front of the variable.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28R3	2. 28R4	3a. 28R5	3b. 28R6	3c. 28R7	3d. 28R8
3e. 28R9	3f. 28RB	3g. 28RC	3h. 28RD	3i. 28RF	3j. 28RG
4. 28RH					

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Quadratic polynomials

Exercise 5 – 2: Quadratic polynomials

1. Solve the following quadratic equations by factorisation. Answers may be left in surd form, where applicable.

a) $7p^2 + 14p = 0$

Solution:

$$\begin{aligned} 7p^2 + 14p &= 0 \\ 7p(p + 2) &= 0 \\ p(p + 2) &= 0 \\ p &= 0 \text{ or } p = -2 \end{aligned}$$

b) $k^2 + 5k - 36 = 0$

Solution:

$$\begin{aligned} k^2 + 5k - 36 &= 0 \\ (k - 4)(k + 9) &= 0 \\ k &= 4 \text{ or } k = -9 \end{aligned}$$

c) $400 = 16h^2$

Solution:

$$
\begin{aligned}
16h^2 - 400 &= 0 \\
16(h^2 - 25) &= 0 \\
(h - 5)(h + 5) &= 0 \\
h &= \pm 5
\end{aligned}
$$

d) $(x - 1)(x + 10) + 24 = 0$

Solution:

$$
\begin{aligned}
(x - 1)(x + 10) + 24 &= 0 \\
x^2 + 9x - 10 + 24 &= 0 \\
x^2 + 9x + 14 &= 0 \\
(x + 7)(x + 2) &= 0 \\
x &= -7 \text{ or } x = -2
\end{aligned}
$$

e) $y^2 - 5ky + 4k^2 = 0$

Solution:

$$
\begin{aligned}
y^2 - 5ky + 4k^2 &= 0 \\
(y - 4k)(y - k) &= 0 \\
y &= 4k \text{ or } y = k
\end{aligned}
$$

2. Solve the following equations by completing the square:

a) $p^2 + 10p - 2 = 0$

Solution:

$$
\begin{aligned}
p^2 + 10p - 2 &= 0 \\
p^2 + 10p &= 2 \\
p^2 + 10p + 25 &= 2 + 25 \\
(p + 5)^2 - 27 &= 0 \\
(p + 5)^2 &= 27 \\
\therefore (p + 5) &= \pm\sqrt{27} \\
\therefore (p + 5) &= -\sqrt{27} \text{ or } (p + 5) = \sqrt{27} \\
\therefore p &= -5 - 3\sqrt{3} \text{ or } p = -5 + 3\sqrt{3}
\end{aligned}
$$

b) $2(6y + y^2) = -4$

Solution:

$$
\begin{aligned}
y^2 + 6y &= -2 \\
y^2 + 6y + 9 &= -2 + 9 \\
(y + 3)^2 &= 7 \\
y + 3 &= \pm\sqrt{7} \\
y &= -3 \pm \sqrt{7}
\end{aligned}
$$

c) $x^2 + 5x + 9 = 0$

Solution:

$$
\begin{aligned}
x^2 + 5x + 9 &= 0 \\
x^2 + 5x &= -9 \\
x^2 + 5x + \frac{25}{4} &= -9 + \frac{25}{4} \\
\left(x + \frac{5}{2}\right)^2 &= -\frac{11}{4} \\
x + \frac{5}{2} &= \pm\sqrt{-\frac{11}{4}}
\end{aligned}
$$

No real solution

d) $f^2 + 30 = 2(10 - 8f)$

Solution:

$$
\begin{aligned}
f^2 + 30 &= 2(10 - 8f) \\
f^2 + 16f + 10 &= 0 \\
f^2 + 16f &= -10 \\
f^2 + 16f + 64 &= -10 + 64 \\
(f + 8)^2 &= 54 \\
f + 8 &= \pm\sqrt{54} \\
f &= -8 \pm \sqrt{9 \times 6} \\
\therefore f &= -8 \pm 3\sqrt{6}
\end{aligned}
$$

e) $3x^2 + 6x - 2 = 0$

Solution:

$$3x^2 + 6x - 2 = 0$$

Divide through by 3 to get leading coefficient of 1

$$
\begin{aligned}
x^2 + 2x &= \frac{2}{3} \\
x^2 + 2x + 1 &= \frac{2}{3} + 1 \\
(x + 1)^2 &= \frac{5}{3} \\
x + 1 &= \pm\sqrt{\frac{5}{3}} \\
x &= -1 \pm \sqrt{\frac{5}{3}}
\end{aligned}
$$

3. Solve the following using the quadratic formula.

a) $3m^2 + m - 4 = 0$

Solution:

$$a = 3; \quad b = 1; \quad c = -4$$

$$\begin{aligned} 3m^2 + m - 4 &= 0 \\ m &= \frac{-(1) \pm \sqrt{1^2 - 4(3)(-4)}}{2(3)} \\ &= \frac{-1 \pm \sqrt{1+48}}{6} \\ &= \frac{-1 \pm \sqrt{49}}{6} \\ &= \frac{-1 \pm 7}{6} \end{aligned}$$

$$m = \frac{-1+7}{6} = \frac{6}{6} = 1 \text{ or } m = \frac{-1-7}{6} = \frac{-8}{6} = -\frac{4}{3}$$

Note: it would be easier and faster to solve by factorisation rather than using the quadratic formula. If the question does not specify which method to use, first try to solve by factorisation.

b) $2t^2 + 6t + 5 = 0$

Solution:

$$\begin{aligned} 2t^2 + 6t + 5 &= 0 \\ t &= \frac{-6 \pm \sqrt{(6)^2 - 4(2)(5)}}{2(2)} \\ &= \frac{-6 \pm \sqrt{36-40}}{4} \\ &= \frac{-6 \pm \sqrt{-4}}{4} \end{aligned}$$

No real solution

c) $y^2 - 4y + 2 = 0$

Solution:

$$\begin{aligned} y^2 - 4y + 2 &= 0 \\ y &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} \\ &= \frac{4 \pm \sqrt{16-8}}{2} \\ &= \frac{4 \pm \sqrt{8}}{2} \\ &= 2 \pm \sqrt{2} \end{aligned}$$

$$\therefore y = 2 + \sqrt{2} \text{ or } y = 2 - \sqrt{2}$$

d) $3f - 2 = -2f^2$

Solution:

$$\begin{aligned} 2f^2 + 3f - 2 &= 0 \\ f &= \frac{-3 \pm \sqrt{3^2 - 4(2)(-2)}}{2(2)} \\ &= \frac{-3 \pm \sqrt{9+16}}{4} \\ &= \frac{-3 \pm \sqrt{25}}{4} \\ &= \frac{-3 \pm 5}{4} \end{aligned}$$

$$\text{therefore } f = \frac{-3+5}{4} = \frac{1}{2} \text{ or } f = \frac{-3-5}{4} = -2$$

4. Factorise the following:

a) $27p^3 - 1$

Solution:

$$27p^3 - 1 = (3p - 1)(9p^2 + 3p + 1)$$

b) $16 + \frac{2}{x^3}$

Solution:

$$\begin{aligned} 16 + \frac{2}{x^3} &= 2\left(8 + \frac{1}{x^3}\right) \\ &= 2\left(2 + \frac{1}{x}\right)\left(4 - \frac{2}{x} + \frac{1}{x^2}\right) \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28RJ	1b. 28RK	1c. 28RM	1d. 28RN	1e. 28RP	2a. 28RQ
2b. 28RR	2c. 28RS	2d. 28RT	2e. 28RV	3a. 28RW	3b. 28RX
3c. 28RY	3d. 28RZ	4a. 28S2	4b. 28S3		

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5.2 Cubic polynomials

Exercise 5 – 3: Cubic polynomials

1. Factorise the following:

a) $p^3 - 1$

Solution:

$$p^3 - 1 = (p - 1)(p^2 + p + 1)$$

b) $t^3 + 27$

Solution:

$$t^3 + 27 = (t + 3)(t^2 + 3t + 9)$$

c) $64 - m^3$

Solution:

$$64 - m^3 = (4 - m)(16 + 4m + m^2)$$

d) $k - 125k^4$

Solution:

$$\begin{aligned} k - 125k^4 &= k(1 - 125k^3) \\ &= k(1 - 5k)(1 + 5k + 25k^2) \end{aligned}$$

e) $8a^6 - b^9$

Solution:

$$\begin{aligned}8a^6 - b^9 &= \left(2a^2\right)^3 - \left(b^3\right)^3 \\ &= \left(2a^2 - b^3\right)\left(4a^4 + 2a^2b^3 + b^6\right)\end{aligned}$$

f) $8 - (p + q)^3$
Solution:

$$\begin{aligned}8 - (p + q)^3 &= [2 - (p + q)][4 + 2(p + q) + (p + q)^2] \\ &= (2 - p - q)(4 + 2p + 2q + p^2 + 2pq + q^2) \\ &= (2 - p - q)(4 + 2p + 2q + p^2 + 2pq + q^2)\end{aligned}$$

2. For each of the following:

a) Use long division to determine the quotient $Q(x)$ and the remainder $R(x)$.
b) Write $a(x)$ in the form $a(x) = b(x) \cdot Q(x) + R(x)$.
c) Check your answer by expanding the brackets to get back to the original cubic polynomial.

a) $a(x) = x^3 + 2x^2 + 3x + 7$ is divided by $(x + 1)$
Solution:

$$\begin{array}{r} x^2 + x + 2 \\ x + 1|\overline{x^3 + 2x^2 + 3x + 7} \\ -\underline{\left(x^3 + x^2\right)} \\ 0 + x^2 + 3x \\ -\underline{\left(x^2 + x\right)} \\ 0 + 2x + 7 \\ -\underline{(2x + 2)} \\ 0 + 5 \end{array}$$

$$\begin{aligned}Q(x) &= x^2 + x + 2 \\ R(x) &= 5 \\ \text{and } a(x) &= b(x) \cdot Q(x) + R(x) \\ \therefore a(x) &= (x + 1)(x^2 + x + 2) + 5\end{aligned}$$

Check:

$$\begin{aligned}(x + 1)(x^2 + x + 2) + 5 &= x^3 + x^2 + 2x + x^2 + x + 2 + 5 \\ &= x^3 + 2x^2 + 3x + 7\end{aligned}$$

b) $a(x) = 1 + 4x^2 - 5x - x^3$ and $b(x) = x + 2$
Solution:
$a(x) = -x^3 + 4x^2 - 5x + 1$ and $b(x) = x + 2$

$$\begin{array}{r} -x^2 + 6x - 17 \\ x + 2|\overline{-x^3 + 4x^2 - 5x + 1} \\ -\underline{\left(-x^3 - 2x^2\right)} \\ 0 + 6x^2 - 5x \\ -\underline{\left(6x^2 + 12x\right)} \\ 0 - 17x + 1 \\ -\underline{(-17x - 34)} \\ 0 + 35 \end{array}$$

$$\begin{aligned} Q(x) &= -x^2 + 6x - 17 \\ R(x) &= 35 \\ \text{and } a(x) &= b(x) \,.\, Q(x) + R(x) \\ \therefore a(x) &= (x+2)(-x^2 + 6x - 17) + 35 \end{aligned}$$

Check:

$$\begin{aligned} (x+2)(-x^2+6x-17)+35 &= -x^3 + 6x^2 - 17x - 2x^2 + 12x - 34 + 35 \\ &= -x^3 + 4x^2 - 5x + 1 \end{aligned}$$

c) $a(x) = 2x^3 + 3x^2 + x - 6$ and $b(x) = x - 1$

Solution:

$$\begin{array}{r} 2x^2 + 5x + 6 \\ x - 1 \overline{|2x^3 + 3x^2 + x - 6} \\ -\left(\underline{2x^3 - 2x^2}\right) \\ 0 + 5x^2 + x \\ -\left(\underline{5x^2 - 5x}\right) \\ 0 + 6x - 6 \\ -\left(\underline{6x - 6}\right) \\ 0 + 0 \end{array}$$

The remainder is equal to zero, therefore $b(x)$ is a factor of $a(x)$.

$$\begin{aligned} Q(x) &= 2x^2 + 5x + 6 \\ R(x) &= 0 \\ \text{and } a(x) &= b(x) \,.\, Q(x) + R(x) \\ \therefore a(x) &= (x-1)(2x^2 + 5x + 6) \end{aligned}$$

Check:

$$\begin{aligned} (x-1)(2x^2+5x+6) &= 2x^3 + 5x^2 + 6x - 2x^2 - 5x - 6 \\ &= 2x^3 + 3x^2 + x - 6 \end{aligned}$$

d) $a(x) = x^3 + 2x^2 + 5$ and $b(x) = x - 1$

Solution:

$$\begin{array}{r} x^2 + 3x + 3 \\ x - 1 \overline{|x^3 + 2x^2 + 5} \\ -\left(\underline{x^3 - x^2}\right) \\ 0 + 3x^2 + 0x \\ -\left(\underline{3x^2 - 3x}\right) \\ 0 + 3x + 5 \\ -\left(\underline{3x - 3}\right) \\ 0 + 8 \end{array}$$

$$\begin{aligned} Q(x) &= x^2 + 3x + 3 \\ R(x) &= 8 \\ \text{and } a(x) &= b(x) \,.\, Q(x) + R(x) \\ \therefore a(x) &= (x-1)(x^2 + 3x + 3) + 8 \end{aligned}$$

Check:

$$(x-1)(x^2+3x+3)+8 = x^3+3x^2+3x-x^2-3x-3+8$$
$$= x^3+2x^2+5$$

e) $(x-1)$ is divided into $a(x) = x^4+2x^3-3x^2+5x+4$

Solution:

$$\begin{array}{r} x^3+3x^2+0x+5 \\ x-1\overline{)x^4+2x^3-3x^2+5x+4} \\ -\underline{(x^4-x^3)} \\ 0+3x^3-3x^2 \\ -\underline{(3x^3-3x^2)} \\ 0+0+5x+4 \\ -\underline{(5x-5)} \\ 0+9 \end{array}$$

$$Q(x) = x^3+3x^2+5$$
$$R(x) = 9$$
and $a(x) = b(x)\ .\ Q(x)+R(x)$
$$\therefore a(x) = (x-1)(x^3+3x^2+5)+9$$

Check:

$$(x-1)(x^3+3x^2+5)+9 = x^4+3x^3+5x-x^3-3x^2-5+9$$
$$= x^4+2x^3-3x^2+5x+4$$

f) $\frac{a(x)}{b(x)} = \frac{5x^4+3x^3+6x^2+x+2}{x^2-2}$

Solution:

$$\begin{array}{r} 5x^2+3x+16 \\ x^2+0x-2\overline{)5x^4+3x^3+6x^2+x+2} \\ -\underline{(5x^4-0x^3-10x^2)} \\ 0+3x^3+16x^2+x \\ -\underline{(3x^3+0x^2-6x)} \\ 0+16x^2+7x+2 \\ -\underline{(16x^2+0x-32)} \\ 7x+34 \end{array}$$

$$Q(x) = 5x^2+3x+16$$
$$R(x) = 7x+34$$
and $a(x) = b(x)\ .\ Q(x)+R(x)$
$$\therefore a(x) = (x^2-2)(5x^2+3x+16)+7x+34$$

Check:

$$(x^2-2)(5x^2+3x+16)+7x+34$$
$$= 5x^4+3x^3+16x^2-10x^2-6x-32+7x+34$$
$$= 5x^4+3x^3+6x^2+x+2$$

g) $a(x) = 3x^3 - x^2 + 2x + 1$ is divided by $(3x - 1)$

Solution:

$$\begin{array}{r} x^2 + \frac{2}{3} \\ 3x-1\overline{)3x^3 - x^2 + 2x + 1} \\ -\underline{(3x^3 - x^2)} \\ 0 + 0 + 2x + 1 \\ -\underline{\left(2x - \frac{2}{3}\right)} \\ 0 + \frac{5}{3} \end{array}$$

$$\begin{aligned} Q(x) &= x^2 + \frac{2}{3} \\ R(x) &= \frac{5}{3} \\ \text{and } a(x) &= b(x) \,.\, Q(x) + R(x) \\ \therefore a(x) &= (3x - 1)\left(x^2 + \frac{2}{3}\right) + \frac{5}{3} \end{aligned}$$

Check:

$$\begin{aligned} (3x-1)\left(x^2 + \frac{2}{3}\right) + \frac{5}{3} &= 3x^3 + 2x - x^2 - \frac{2}{3} + \frac{5}{3} \\ &= 3x^3 - x^2 + 2x + 1 \end{aligned}$$

h) $a(x) = 2x^5 + x^3 + 3x^2 - 4$ and $b(x) = x + 2$

Solution:

$$\begin{array}{r} 2x^4 - 4x^3 + 9x^2 - 15x + 30 \\ x+2\overline{)2x^5 + 0x^4 + x^3 + 3x^2 + 0x - 4} \\ -\underline{(2x^5 + 4x^4)} \\ 0 - 4x^4 + x^3 \\ -\underline{(-4x^4 - 8x^3)} \\ 0 + 9x^3 + 3x^2 \\ -\underline{(9x^3 + 18x^2)} \\ 0 - 15x^2 + 0x \\ -\underline{(-15x^2 - 30x)} \\ 0 + 30x - 4 \\ -\underline{(30x + 60)} \\ 0 - 64 \end{array}$$

$$\begin{aligned} Q(x) &= 2x^4 - 4x^3 + 9x^2 - 15x + 30 \\ R(x) &= -64 \\ \text{and } a(x) &= b(x) \,.\, Q(x) + R(x) \\ \therefore a(x) &= (x+2)(2x^4 - 4x^3 + 9x^2 - 15x + 30) - 64 \end{aligned}$$

Check:

$$\begin{aligned} &(x+2)(2x^4 - 4x^3 + 9x^2 - 15x + 30) - 64 \\ &= 2x^5 - 4x^4 + 9x^3 - 15x^2 + 30x + 4x^4 - 8x^3 + 18x^2 - 30x + 60 - 64 \\ &= 2x^5 + x^3 + 3x^2 - 4 \end{aligned}$$

3. Use synthetic division to determine the quotient $Q(x)$ and the remainder $R(x)$ when $f(x)$ is divided by $g(x)$:

a)

$$f(x) = x^2 + 5x + 1$$
$$g(x) = x + 2$$

Solution:

$$\begin{array}{r|lll} & 1 & 3 & -5 \\ \hline -2 & 1 & 5 & 1 \end{array}$$

$$q_1 = 1$$
$$q_0 = 5 + (-2)(1) = 3$$
$$R = 1 + (-2)(3) = -5$$

$$Q(x) = x + 3$$
$$R(x) = -5$$
and $f(x) = g(x) \,.\, Q(x) + R(x)$
$$\therefore f(x) = (x + 2)(x + 3) - 5$$

b)

$$f(x) = x^2 - 5x - 7$$
$$g(x) = x - 1$$

Solution:

$$\begin{array}{r|lll} & 1 & -4 & -11 \\ \hline 1 & 1 & -5 & -7 \end{array}$$

$$q_1 = 1$$
$$q_0 = -5 + (1)(1) = -4$$
$$R = -7 + (-4)(1) = -11$$

$$Q(x) = x - 4$$
$$R(x) = -11$$
and $f(x) = g(x) \,.\, Q(x) + R(x)$
$$\therefore f(x) = (x - 1)(x - 4) - 11$$

c)

$$f(x) = 2x^3 + 5x - 4$$
$$g(x) = x - 1$$

Solution:

$$\begin{array}{r|llll} & 2 & 2 & 7 & 3 \\ \hline 1 & 2 & 0 & 5 & -4 \end{array}$$

$$q_2 = 2$$
$$q_1 = 0 + (2)(1) = 2$$
$$q_0 = 5 + (2)(1) = 7$$
$$R = -4 + (7)(1) = 3$$

$$Q(x) = 2x^2 + 2x + 7$$
$$R(x) = 3$$
and $f(x) = g(x) \,.\, Q(x) + R(x)$
$$\therefore f(x) = (x - 1)(2x^2 + 2x + 7) + 3$$

d)

$$\begin{aligned} f(x) &= 19 + x^2 + 8x \\ g(x) &= x + 3 \end{aligned}$$

Solution:

$$\begin{array}{r|lll} & 1 & 5 & 4 \\ -3 & 1 & 8 & 19 \end{array}$$

$$\begin{aligned} q_1 &= 1 \\ q_0 &= 8 + (-3)(1) = 5 \\ R &= 19 + (5)(-3) = 4 \end{aligned}$$

$$\begin{aligned} Q(x) &= x + 5 \\ R(x) &= 4 \\ \text{and } f(x) &= g(x)\ .\ Q(x) + R(x) \\ \therefore f(x) &= (x + 3)(x + 5) + 4 \end{aligned}$$

e)

$$\begin{aligned} f(x) &= x^3 + 2x^2 + x - 10 \\ g(x) &= x - 1 \end{aligned}$$

Solution:

$$\begin{array}{r|llll} & 1 & 3 & 4 & -6 \\ 1 & 1 & 2 & 1 & -10 \end{array}$$

$$\begin{aligned} q_2 &= 1 \\ q_1 &= 2 + (1)(1) = 3 \\ q_0 &= 1 + (1)(3) = 4 \\ R &= -10 + (1)(4) = -6 \end{aligned}$$

$$\begin{aligned} Q(x) &= x^2 + 3x + 4 \\ R(x) &= -6 \\ \text{and } f(x) &= g(x)\ .\ Q(x) + R(x) \\ \therefore f(x) &= (x - 1)(x^2 + 3x + 4) - 6 \end{aligned}$$

f)

$$\begin{aligned} f(x) &= 2x^3 + 7x^2 + 2x - 3 \\ g(x) &= x + 3 \end{aligned}$$

Solution:

$$\begin{array}{r|llll} & 2 & 1 & -1 & +0 \\ -3 & 2 & 7 & 2 & -3 \end{array}$$

$$\begin{aligned} q_2 &= 2 \\ q_1 &= 7 + (-3)(2) = 1 \\ q_0 &= 2 + (-3)(1) = -1 \\ R &= -3 + (-3)(-1) = 0 \end{aligned}$$

$$\begin{aligned} Q(x) &= 2x^2 + x - 1 \\ R(x) &= 0 \\ \text{and } f(x) &= g(x)\ .\ Q(x) + R(x) \\ \therefore f(x) &= (x + 3)(2x^2 + x - 1) \end{aligned}$$

g)

$$\begin{aligned} f(x) &= 4x^3 + 4x^2 - x - 2 \\ g(x) &= 2x - 1 \\ &= 2\left(x - \frac{1}{2}\right) \end{aligned}$$

Solution:

$$\begin{array}{l|llll} & 4 & 6 & 2 & -1 \\ \hline \frac{1}{2} & 4 & 4 & -1 & -2 \end{array}$$

$$\begin{aligned} q_2 &= 4 \\ q_1 &= 4 + \left(\frac{1}{2}\right)(4) = 6 \\ q_0 &= -1 + \left(\frac{1}{2}\right)(6) = 2 \\ R &= -2 + \left(\frac{1}{2}\right)(2) = -1 \end{aligned}$$

$$\begin{aligned} Q(x) &= 4x^2 + 6x + 2 \\ R(x) &= -1 \\ \text{and } f(x) &= \frac{1}{2}g(x)\ .\ Q(x) + R(x) \\ &= \frac{1}{2}\ .\ 2\left(x - \frac{1}{2}\right)(4x^2 + 6x + 2) - 1 \\ &= \left(x - \frac{1}{2}\right)(2)(2x^2 + 3x + 1) - 1 \\ \therefore f(x) &= (2x - 1)(2x^2 + 3x + 1) - 1 \end{aligned}$$

h)

$$\begin{aligned} f(x) &= 5x + 22 + 2x^3 + x^2 \\ g(x) &= 2x + 3 \\ &= 2\left(x + \frac{3}{2}\right) \end{aligned}$$

Solution:

$$\begin{aligned} f(x) &= 2x^3 + x^2 + 5x + 22 \\ g(x) &= 2x + 3 \end{aligned}$$

$$\begin{array}{l|llll} & 2 & -2 & 8 & 10 \\ \hline -\frac{3}{2} & 2 & 1 & 5 & 22 \end{array}$$

$$\begin{aligned} q_2 &= 2 \\ q_1 &= 1 + \left(-\frac{3}{2}\right)(2) = -2 \\ q_0 &= 5 + \left(-\frac{3}{2}\right)(-2) = 8 \\ R &= 22 + \left(-\frac{3}{2}\right)(8) = 10 \end{aligned}$$

$$
\begin{aligned}
Q(x) &= 2x^2 - 2x + 8 \\
R(x) &= 10 \\
\text{and } f(x) &= \frac{1}{2}g(x) \,.\, Q(x) + R(x) \\
&= \frac{1}{2} \,.\, 2\left(x + \frac{3}{2}\right)(2x^2 - 2x + 8) + 10 \\
&= 2\left(x + \frac{3}{2}\right)(x^2 - x + 4) + 10 \\
\therefore f(x) &= (2x + 3)(x^2 - x + 4) + 10
\end{aligned}
$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28S7 1b. 28S8 1c. 28S9 1d. 28SB 1e. 28SC 1f. 28SD
2a. 28SF 2b. 28SG 2c. 28SH 2d. 28SJ 2e. 28SK 2f. 28SM
2g. 28SN 2h. 28SP 3a. 28SQ 3b. 28SR 3c. 28SS 3d. 28ST
3e. 28SV 3f. 28SW 3g. 28SX 3h. 28SY

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5.3 Remainder theorem

Exercise 5 – 4: Remainder theorem

1. Use the remainder theorem to determine the remainder R when $g(x)$ is divided by $h(x)$:

a)

$$
\begin{aligned}
g(x) &= x^3 + 4x^2 + 11x - 5 \\
h(x) &= x - 1
\end{aligned}
$$

Solution:

$$
\begin{aligned}
g(x) &= x^3 + 4x^2 + 11x - 5 \\
g(1) &= (1)^3 + 4(1)^2 + 11(1) - 5 \\
&= 1 + 4 + 11 - 5 \\
\therefore R &= 11
\end{aligned}
$$

b)

$$
\begin{aligned}
g(x) &= 2x^3 - 5x^2 + 8 \\
h(x) &= 2x - 1
\end{aligned}
$$

Solution:

$$
\begin{aligned}
g(x) &= 2x^3 - 5x^2 + 8 \\
g\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^3 - 5\left(\frac{1}{2}\right)^2 + 8 \\
&= 2\left(\frac{1}{8}\right) - 5\left(\frac{1}{4}\right) + 8 \\
&= \frac{1}{4} - \frac{5}{4} + 8 \\
\therefore R &= 7
\end{aligned}
$$

c)

$$\begin{aligned} g(x) &= 4x^3 + 5x^2 + 6x - 1 \\ h(x) &= x + 2 \end{aligned}$$

Solution:

$$\begin{aligned} g(x) &= 4x^3 + 5x^2 + 6x - 1 \\ g(-2) &= 4(-2)^3 + 5(-2)^2 + 6(-2) - 1 \\ &= -32 + 20 - 12 - 1 \\ \therefore R &= -25 \end{aligned}$$

d)

$$\begin{aligned} g(x) &= -5x^3 - x^2 - 10x + 9 \\ h(x) &= 5x + 1 \end{aligned}$$

Solution:

$$\begin{aligned} g(x) &= -5x^3 - x^2 - 10x + 9 \\ g\left(-\frac{1}{5}\right) &= -5\left(-\frac{1}{5}\right)^3 - \left(-\frac{1}{5}\right)^2 - 10\left(-\frac{1}{5}\right) + 9 \\ &= -5\left(-\frac{1}{125}\right) - \left(\frac{1}{25}\right) + 2 + 9 \\ &= \frac{1}{25} - \frac{1}{25} + 11 \\ \therefore R &= 11 \end{aligned}$$

e)

$$\begin{aligned} g(x) &= x^4 + 5x^2 + 2x - 8 \\ h(x) &= x + 1 \end{aligned}$$

Solution:

$$\begin{aligned} g(x) &= x^4 + 5x^2 + 2x - 8 \\ g(-1) &= (-1)^4 + 5(-1)^2 + 2(-1) - 8 \\ &= 1 + 5 - 2 - 8 \\ &= 6 - 10 \\ \therefore R &= -4 \end{aligned}$$

f)

$$\begin{aligned} g(x) &= 3x^5 - 8x^4 + x^2 + 2 \\ h(x) &= 2 - x \end{aligned}$$

Solution:

$$\begin{aligned} h(x) &= 2 - x \\ \text{If } x = 2: \quad h(2) &= 2 - 2 = 0 \\ g(x) &= h(x) \,.\, Q(x) + R \\ g(2) &= h(2) \,.\, Q(2) + R \\ \therefore g(2) &= 0 \,.\, Q(2) + R \\ g(2) &= 0 + R \\ \therefore R &= 3(2)^5 - 8(2)^4 + (2)^2 + 2 \\ &= 3(32) - 8(16) + 6 \\ &= 96 - 128 + 6 \\ \therefore R &= -26 \end{aligned}$$

g)

$$g(x) = 2x^{100} - x - 1$$
$$h(x) = x + 1$$

Solution:

$$\begin{aligned} g(x) &= 2x^{100} - x - 1 \\ g(-1) &= 2(-1)^{100} - (-1) - 1 \\ &= 2 + 1 - 1 \\ \therefore R &= 2 \end{aligned}$$

2. Determine the value of t if $x^3 + tx^2 + 8x + 21$ divided by $x + 1$ gives a remainder of 16.

   Solution:

$$\begin{aligned} a(x) &= x^3 + tx^2 + 8x + 21 \\ b(x) &= x + 1 \\ R &= 16 \\ a(-1) &= (-1)^3 + t(-1)^2 + 8(-1) + 21 \\ \therefore 16 &= -1 + t - 8 + 21 \\ \therefore t &= 4 \end{aligned}$$

3. Calculate the value of m if $2x^3 - 7x^2 + mx - 26$ is divided by $x - 2$ and gives a remainder of -24.

   Solution:

$$\begin{aligned} a(x) &= 2x^3 - 7x^2 + mx - 26 \\ b(x) &= x - 2 \\ R &= -24 \\ a(2) &= 2(2)^3 - 7(2)^2 + m(2) - 26 \\ \therefore -24 &= 16 - 28 + 2m - 26 \\ 14 &= 2m \\ \therefore m &= 7 \end{aligned}$$

4. If $x^5 - 2x^3 - kx - 1$ is divided by $x - 1$ and the remainder is $-\frac{1}{2}$, find the value of k .

   Solution:

$$\begin{aligned} a(x) &= x^5 - 2x^3 - kx - 1 \\ b(x) &= x - 1 \\ R &= -\frac{1}{2} \\ a(1) &= (1)^5 - 2(1)^3 - k(1) - 1 \\ \therefore -\frac{1}{2} &= 1 - 2 - k - 1 \\ \frac{3}{2} &= -k \\ \therefore k &= -\frac{3}{2} \end{aligned}$$

5. Determine the value of p if $18x^3 + px^2 - 8x + 9$ is divided by $2x - 1$ and gives a remainder of 6.

Solution:

$$\begin{aligned}
a(x) &= 18x^3 + px^2 - 8x + 9 \\
b(x) &= 2x - 1 \\
R &= 6 \\
a\left(\frac{1}{2}\right) &= 18\left(\frac{1}{2}\right)^3 + p\left(\frac{1}{2}\right)^2 - 8\left(\frac{1}{2}\right) + 9 \\
\therefore 6 &= 18\left(\frac{1}{8}\right) + p\left(\frac{1}{4}\right) - 4 + 9 \\
6 &= \frac{18}{8} + \frac{p}{4} + 5 \\
1 &= \frac{18}{8} + \frac{p}{4} \\
4 &= 9 + p \\
\therefore p &= -5
\end{aligned}$$

6. If $x^3 + x^2 - x + b$ is divided by $x - 2$ and the remainder is $2\frac{1}{2}$, calculate the value of b.

   Solution:

$$\begin{aligned}
\text{Let } f(x) &= x^3 + x^2 - x + b \\
R &= \frac{5}{2} \\
f(2) &= (2)^3 + (2)^2 - (2) + b \\
\therefore \frac{5}{2} &= 8 + 4 - 2 + b \\
\frac{5}{2} - 10 &= b \\
\therefore b &= -\frac{15}{2}
\end{aligned}$$

7. Calculate the value of h if $3x^5 + hx^4 + 10x^2 - 21x + 12$ is divided by $x - 2$ and gives a remainder of 10.

   Solution:

$$\begin{aligned}
a(x) &= 3x^5 + hx^4 + 10x^2 - 21x + 12 \\
b(x) &= x - 2 \\
R &= 10 \\
a(2) &= 3(2)^5 + h(2)^4 + 10(2)^2 - 21(2) + 12 \\
\therefore 10 &= 3(32) + 16h + 40 - 42 + 12 \\
10 &= 96 + 16h + 10 \\
-96 &= 16h \\
\therefore h &= -6
\end{aligned}$$

8. If $x^3 + 8x^2 + mx - 5$ is divided by $x + 1$ and the remainder is n, express m in terms of n.

   Solution:

$$\begin{aligned}
a(x) &= x^3 + 8x^2 + mx - 5 \\
b(x) &= x + 1 \\
R &= n \\
a(-1) &= (-1)^3 + 8(-1)^2 + m(-1) - 5 \\
\therefore n &= -1 + 8 - m - 5 \\
n &= 2 - m \\
\therefore m &= 2 - n
\end{aligned}$$

9. When the polynomial $2x^3 + px^2 + qx + 1$ is divided by $x + 1$ or $x - 4$, the remainder is 5. Determine the values of p and q.

Solution:

$$\begin{aligned} a(x) &= 2x^3 + px^2 + qx + 1 \\ a(-1) &= 2(-1)^3 + p(-1)^2 + q(-1) + 1 \\ \therefore 5 &= -2 + p - q + 1 \\ 6 &= p - q \\ \therefore q &= p - 6 \ldots\ldots (1) \\ a(4) &= 2(4)^3 + p(4)^2 + q(4) + 1 \\ \therefore 5 &= 128 + 16p + 4q + 1 \\ -124 &= 16p + 4q \\ \therefore -31 &= 4p + q \ldots\ldots (2) \\ \text{Substitute eqn } (1) \text{ into eqn } (2) \\ \therefore -31 &= 4p + (p - 6) \\ -25 &= 5p \\ \therefore -5 &= p \\ \text{And } q &= -5 - 6 \\ &= -11 \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28SZ	1b. 28T2	1c. 28T3	1d. 28T4	1e. 28T5	1f. 28T6
1g. 28T7	2. 28T8	3. 28T9	4. 28TB	5. 28TC	6. 28TD
7. 28TF	8. 28TG	9. 28TH			

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5.4 Factor theorem

Exercise 5 – 5: Factorising cubic polynomials

1. Find the remainder when $4x^3 - 4x^2 + x - 5$ is divided by $x + 1$.

Solution:

$$\begin{aligned} \text{Let } a(x) &= 4x^3 - 4x^2 + x - 5 \\ a(-1) &= 4(-1)^3 - 4(-1)^2 + (-1) - 5 \\ &= -4 - 4 - 1 - 5 \\ &= -14 \end{aligned}$$

2. Use the factor theorem to factorise $x^3 - 3x^2 + 4$ completely.

Solution:

$$\begin{aligned}
\text{Let } a(x) &= x^3 - 3x^2 + 4 \\
a(-1) &= (-1)^3 - 3(-1)^2 + 4 \\
&= -1 - 3 + 4 \\
&= 0 \\
\therefore (x+1) &\text{ is a factor} \\
a(x) &= (x+1)(x^2 - 4x + 4) \\
&= (x+1)(x-2)(x-2) \\
&= (x+1)(x-2)^2
\end{aligned}$$

3. $f(x) = 2x^3 + x^2 - 5x + 2$

   a) Find $f(1)$.

   Solution:

$$\begin{aligned}
f(x) &= 2x^3 + x^2 - 5x + 2 \\
f(1) &= 2(1)^3 + (1)^2 - 5(1) + 2 \\
&= 2 + 1 - 5 + 2 \\
&= 0
\end{aligned}$$

   b) Factorise $f(x)$ completely.

   Solution:

$$\begin{aligned}
f(1) &= 0 \\
\therefore (x-1) &\text{ is a factor of } f(x) \\
f(x) &= (x-1)(2x^2 + 3x - 2) \\
&= (x-1)(2x-1)(x+2)
\end{aligned}$$

4. Use the factor theorem to determine all the factors of the following expression: $x^3 + x^2 - 17x + 15$

   Solution:

$$\begin{aligned}
\text{Let } a(x) &= x^3 + x^2 - 17x + 15 \\
a(1) &= (1)^3 + (1)^2 - 17(1) + 15 \\
&= 1 + 1 - 17 + 15 \\
&= 0 \\
\therefore a(x) &= (x-1)(x^2 + 2x - 15) \\
&= (x-1)(x+5)(x-3)
\end{aligned}$$

5. Complete: If $f(x)$ is a polynomial and p is a number such that $f(p) = 0$, then $(x - p)$ is...

   Solution:

   a factor of $f(x)$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28TJ 2. 28TK 3a. 28TM 3b. 28TN 4. 28TP 5. 28TQ

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Exercise 5 – 6: Solving cubic equations

Solve the following cubic equations:

1. $x^3 + x^2 - 16x = 16$

   Solution:

$$\begin{aligned} x^3 + x^2 - 16x &= 16 \\ x^3 + x^2 - 16x - 16 &= 0 \\ \text{Let } a(x) &= x^3 + x^2 - 16x - 16 \\ a(-1) &= (-1)^3 + (-1)^2 - 16(-1) - 16 \\ &= -1 + 1 + 16 - 16 \\ &= 0 \\ \therefore a(x) &= (x+1)(x^2 - 16) \\ &= (x+1)(x-4)(x+4) \\ \therefore 0 &= (x+1)(x-4)(x+4) \\ \therefore x = -1 \text{ or } x &= 4 \text{ or } x = -4 \end{aligned}$$

2. $-n^3 - n^2 + 22n + 40 = 0$

   Solution:

$$\begin{aligned} n^3 + n^2 - 22n - 40 &= 0 \\ \text{Let } a(n) &= n^3 + n^2 - 22n - 40 \\ a(-2) &= (-2)^3 + (-2)^2 - 22(-2) - 40 \\ &= -8 + 4 + 44 - 40 \\ &= 0 \\ \therefore a(n) &= (n+2)(n^2 - n - 20) \\ &= (n+2)(n-5)(n+4) \\ \therefore 0 &= (n+2)(n-5)(n+4) \\ \therefore n = -2 \text{ or } n &= -4 \text{ or } n = 5 \end{aligned}$$

3. $y(y^2 + 2y) = 19y + 20$

   Solution:

$$\begin{aligned} y(y^2 + 2y) &= 19y + 20 \\ y^3 + 2y^2 - 19y - 20 &= 0 \\ \text{Let } a(y) &= y^3 + 2y^2 - 19y - 20 \\ a(-1) &= (-1)^3 + 2(-1)^2 - 19(-1) - 20 \\ &= -1 + 2 + 19 - 20 \\ &= 0 \\ \therefore a(y) &= (y+1)(y^2 + y - 20) \\ &= (y+1)(y+5)(y-4) \\ \therefore 0 &= (y+1)(y+5)(y-4) \\ \therefore y = -1 \text{ or } y &= 4 \text{ or } y = -5 \end{aligned}$$

4. $k^3 + 9k^2 + 26k + 24 = 0$

Solution:

$$\begin{aligned}
\text{Let } a(k) &= k^3 + 9k^2 + 26k + 24 \\
a(-2) &= (-2)^3 + 9(-2)^2 + 26(-2) + 24 \\
&= -8 + 36 - 52 + 24 \\
&= 0 \\
\therefore a(k) &= (k+2)(k^2 + 7k + 12) \\
&= (k+2)(k+3)(k+4) \\
\therefore 0 &= (k+2)(k+3)(k+4) \\
\therefore k &= -2 \text{ or } k = -3 \text{ or } k = -4
\end{aligned}$$

5. $x^3 + 2x^2 - 50 = 25x$

Solution:

$$\begin{aligned}
x^3 + 2x^2 - 50 &= 25x \\
x^3 + 2x^2 - 25x - 50 &= 0 \\
\text{Let } a(x) &= x^3 + 2x^2 - 25x - 50 \\
a(-2) &= (-2)^3 + 2(-2)^2 - 25(-2) - 50 \\
&= -8 + 8 + 50 - 50 \\
&= 0 \\
\therefore a(x) &= (x+2)(x^2 - 25) \\
&= (x+2)(x-5)(x+5) \\
\therefore 0 &= (x+2)(x-5)(x+5) \\
\therefore x &= -2 \text{ or } x = 5 \text{ or } x = -5
\end{aligned}$$

6. $-p^3 + 19p = 30$

Solution:

$$\begin{aligned}
-p^3 + 19p - 30 &= 0 \\
p^3 - 19p + 30 &= 0 \\
\text{Let } a(p) &= p^3 - 19p + 30 \\
a(3) &= (3)^3 - 19(3) + 30 \\
&= 27 - 57 + 30 \\
&= 0 \\
\therefore a(p) &= (p-3)(p^2 + 3p - 10) \\
&= (p-3)(p-2)(p+5) \\
\therefore 0 &= (p-3)(p-2)(p+5) \\
\therefore p &= 3 \text{ or } p = 2 \text{ or } p = -5
\end{aligned}$$

7. $6x^2 - x^3 = 5x + 12$

Solution:

$$\begin{aligned}
0 &= x^3 - 6x^2 + 5x + 12 \\
\text{Let } a(x) &= x^3 - 6x^2 + 5x + 12 \\
a(-1) &= (-1)^3 - 6(-1)^2 + 5(-1) + 12 \\
&= -1 - 6 - 5 + 12 \\
&= 0 \\
\therefore a(x) &= (x+1)(x^2 - 7x + 12) \\
&= (x+1)(x-3)(x-4) \\
\therefore 0 &= (x+1)(x-3)(x-4) \\
\therefore x &= -1 \text{ or } x = 3 \text{ or } x = 4
\end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.
1. 28TR 2. 28TS 3. 28TT 4. 28TV 5. 28TW 6. 28TX
7. 28TY

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5.6 Summary

Exercise 5 – 7: End of chapter exercises

1. Solve for x: $x^3 + x^2 - 5x + 3 = 0$

   Solution:

$$\begin{aligned} \text{Let } a(x) &= x^3 + x^2 - 5x + 3 \\ a(1) &= (1)^3 + (1)^2 - 5(1) + 3 \\ &= 0 \\ \therefore a(x) &= (x-1)(x^2 + 2x - 3) \\ &= (x-1)(x+3)(x-1) \\ &= (x-1)^2(x+3) \\ \therefore 0 &= (x-1)^2(x+3) \\ \therefore x &= 1 \text{ or } x = -3 \end{aligned}$$

2. Solve for y: $y^3 = 3y^2 + 16y + 12$

   Solution:

$$\begin{aligned} \text{Let } a(y) &= y^3 - 3y^2 - 16y - 12 \\ a(-1) &= (-1)^3 - 3(-1)^2 - 16(-1) - 12 \\ &= -1 - 3 + 16 - 12 \\ &= 0 \\ \therefore a(y) &= (y+1)(y^2 - 4y - 12) \\ &= (y+1)(y-6)(y+2) \\ \therefore 0 &= (y+1)(y-6)(y+2) \\ \therefore y &= -1 \text{ or } y = 6 \text{ or } y = -2 \end{aligned}$$

3. Solve for m: $m(m^2 - m - 4) = -4$

   Solution:

$$\begin{aligned} \text{Let } a(m) &= m^3 - m^2 - 4m + 4 \\ a(1) &= (1)^3 - (1)^2 - 4(1) + 4 \\ &= 1 - 1 - 4 + 4 \\ &= 0 \\ \therefore a(m) &= (m-1)(m^2 - 4) \\ &= (m-1)(m+2)(m-2) \\ \therefore 0 &= (m-1)(m+2)(m-2) \\ \therefore m &= 1 \text{ or } m = 2 \text{ or } m = -2 \end{aligned}$$

4. Solve for x: $x^3 - x^2 = 3(3x + 2)$

Solution:

$$
\begin{aligned}
x^3 - x^2 &= 3(3x+2) \\
x^3 - x^2 &= 9x + 6 \\
x^3 - x^2 - 9x - 6 &= 0 \\
\text{Let } x = -2: \quad (-2)^3 - (-2)^2 - 9(-2) - 6 & \\
&= -8 - 4 + 18 - 6 \\
&= 0 \\
\therefore (x+2) \text{ is a factor} & \\
(x+2)(x^2 - 3x - 3) &= 0 \\
\text{Using quadratic formula to solve for } x: x^2 - 3x - 3 &= 0 \\
a = 1; \quad b = -3; \quad c &= -3 \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-3)}}{2(1)} \\
&= \frac{3 \pm \sqrt{9 + 12}}{2} \\
&= \frac{3 \pm \sqrt{21}}{2} \\
\therefore x = -2 \text{ or } x &= \frac{3 + \sqrt{21}}{2} \text{ or } x = \frac{3 - \sqrt{21}}{2}
\end{aligned}
$$

5. Solve for x if $2x^3 - 3x^2 - 8x = 3$.

Solution:

$$
\begin{aligned}
2x^3 - 3x^2 - 8x &= 3 \\
2x^3 - 3x^2 - 8x - 3 &= 0 \\
\text{Let } a(x) &= 2x^3 - 3x^2 - 8x - 3 \\
a(-1) &= 2(-1)^3 - 3(-1)^2 - 8(-1) - 3 \\
&= -2 - 3 + 8 - 3 \\
&= 0 \\
\therefore a(x) &= (x+1)(2x^2 - 5x - 3) \\
&= (x+1)(2x+1)(x-3) \\
\therefore 0 &= (x+1)(2x+1)(x-3) \\
\therefore x = -1 \text{ or } x &= -\frac{1}{2} \text{ or } x = 3
\end{aligned}
$$

6. Solve for x: $16(x + 1) = x^2(x + 1)$

Solution:

$$
\begin{aligned}
16(x+1) &= x^2(x+1) \\
16x+16 &= x^3+x^2 \\
0 &= x^3+x^2-16x-16 \\
\text{Let } a(x) &= x^3+x^2-16x-16 \\
a(-1) &= (-1)^3+(-1)^2-16(-1)-16 \\
&= -1+1+16-16 \\
&= 0 \\
\therefore a(x) &= (x+1)(x^2-16) \\
&= (x+1)(x-4)(x+4) \\
\therefore 0 &= (x+1)(x-4)(x+4) \\
\therefore x &= -1 \text{ or } x=4 \text{ or } x=-4
\end{aligned}
$$

7. a) Show that $x-2$ is a factor of $3x^3-11x^2+12x-4$.

Solution:

$$
\begin{aligned}
\text{Let } a(x) &= 3x^3-11x^2+12x-4 \\
a(2) &= 3(2)^3-11(2)^2+12(2)-4 \\
&= 24-44+24-4 \\
&= 0 \\
\therefore (x-2) &\text{ is a factor of } a(x)
\end{aligned}
$$

b) Hence, by factorising completely, solve the equation:
$3x^3-11x^2+12x-4=0$

Solution:

$$
\begin{aligned}
3x^3-11x^2+12x-4 &= 0 \\
(x-2)(3x^2-5x+2) &= 0 \\
\therefore (x-2)(3x-2)(x-1) &= 0 \\
\therefore x = 2 \text{ or } x = \frac{2}{3} &\text{ or } x=1
\end{aligned}
$$

8. $2x^3-x^2-2x+2 = Q(x).(2x-1)+R$ for all values of x. What is the value of R?

Solution:

$$
\begin{aligned}
\text{Let } a(x) &= 2x^3-x^2-2x+2 \\
R = a\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^3-\left(\frac{1}{2}\right)^2-2\left(\frac{1}{2}\right)+2 \\
&= 2\left(\frac{1}{8}\right)-\left(\frac{1}{4}\right)-1+2 \\
&= \frac{1}{4}-\frac{1}{4}+1 \\
&= 1 \\
\therefore R &= 1
\end{aligned}
$$

9. a) Use the factor theorem to solve the following equation for m:
$8m^3+7m^2-17m+2=0$

Solution:

$$\text{Let } a(m) = 8m^3 + 7m^2 - 17m + 2$$
$$a(1) = 8(1)^3 + 7(1)^2 - 17(1) + 2$$
$$= 8 + 7 - 17 + 2$$
$$= 0$$
$$\therefore a(m) = (m-1)(8m^2 + 15m - 2)$$
$$= (m-1)(8m-1)(m+2)$$
$$\therefore 0 = (m-1)(8m-1)(m+2)$$
$$\therefore m = 1 \text{ or } m = \frac{1}{8} \text{ or } m = -2$$

b) Hence, or otherwise, solve for x:
$2^{3x+3} + 7.2^{2x} + 2 = 17.2^x$

Solution:

$$2^{3x+3} + 7 \,.\, 2^{2x} + 2 = 17 \,.\, 2^x$$
$$2^{3x} \,.\, 2^3 + 7 \,.\, 2^{2x} + 2 = 17 \,.\, 2^x$$
$$8 \,.\, (2^x)^3 + 7 \,.\, (2^x)^2 - 17 \,.\, 2^x + 2 = 0$$
$$\text{which we can compare with } a(m) = 8m^3 + 7m^2 - 17m + 2$$
$$\text{Let } 2^x = m$$
$$\text{and from part (a) we know that } m = 1 \text{ or } m = \frac{1}{8} \text{ or } m = -2$$
$$\text{So } 2^x = 1$$
$$2^x = 2^0$$
$$\therefore x = 0$$
$$\text{Or } 2^x = \frac{1}{8}$$
$$2^x = 2^{-3}$$
$$\therefore x = -3$$
$$\text{Or } 2^x = -2$$
$$\therefore \text{no solution}$$

10. Find the value of R if $x - 1$ is a factor of $h(x) = (x-6) \,.\, Q(x) + R$ and $Q(x)$ divided by $x - 1$ gives a remainder of 8.

Solution:

$$h(x) = (x-6) \,.\, Q(x) + R$$
$$h(1) = (1-6) \,.\, Q(1) + R$$
$$\therefore 0 = -5 \,.\, Q(1) + R$$
$$\text{And } Q(1) = 8$$
$$0 = -5(8) + R$$
$$\therefore R = 40$$

11. Determine the values of p for which the function

$$f(x) = 3p^3 - (3p-7)\,x^2 + 5x - 3$$

leaves a remainder of 9 when it is divided by $(x - p)$.

Solution:

$$
\begin{aligned}
f(x) &= 3p^3 - (3p-7)x^2 + 5x - 3 \\
\therefore f(p) &= 3p^3 - (3p-7)p^2 + 5p - 3 \\
&= 3p^3 - 3p^3 + 7p^2 + 5p - 3 \\
&= 7p^2 + 5p - 3 \\
f(p) &= 9 \\
\therefore 9 &= 7p^2 + 5p - 3 \\
0 &= 7p^2 + 5p - 12 \\
0 &= (7p+12)(p-1) \\
\therefore p &= -\frac{12}{7} \text{ or } p = 1
\end{aligned}
$$

Alternative (long) method:
We first take out the factor using long division:

$$
\begin{array}{rl}
& \underline{(7-3p)x + (5+7p-3p^2)} \\
(x-p) | & (7-3p)x^2 + 5x + (3p^3-3) \\
& -\underline{\{(7-3p)x^2 - p(7-3p)x\}} \\
& \quad 0 + 5x + p(7-3p)x + (3p^3-3) \\
& \quad 5x + 7px - 3p^2x + 3p^3x \\
& \quad [5+7p-3p^2]x + 3p^3 - 3 \\
& \quad -\underline{\{[5+7p-3p^2]x - p(5+7p-3p^2)\}} \\
& \quad 0 + 3p^3 - 3 + 5p + 7p^2 - 3p^3
\end{array}
$$

We take the remainder and set it equal to 9:

$$
\begin{aligned}
-3 + 5p + 7p^2 &= 9 \\
7p^2 + 5p - 12 &= 0 \\
(7p+12)(p-1) &= 0 \\
\therefore p &= -\frac{12}{7} \text{ or } p = 1
\end{aligned}
$$

12. Calculate t and $Q(x)$ if $x^2 + tx + 3 = (x+4)\,.\,Q(x) - 17$.

Solution:

$$
\begin{aligned}
x^2 + tx + 20 &= (x+4)\,.\,Q(x) \\
\text{Let } f(x) &= x^2 + tx + 20 \\
f(-4) &= (-4)^2 + t(-4) + 20 \\
0 &= 16 - 4t + 20 \\
4t &= 36 \\
\therefore t &= 9
\end{aligned}
$$

$$
\begin{aligned}
x^2 + 9x + 20 &= (x+4)\,.\,Q(x) \\
(x+4)(x+5) &= (x+4)\,.\,Q(x) \\
\therefore Q(x) &= x+5
\end{aligned}
$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28V2	2. 28V3	3. 28V4	4. 28V5	5. 28V6	6. 28V7
7. 28V8	8. 28V9	9. 28VB	10. 28VC	11. 28VD	12. 28VF

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CHAPTER 6

Differential calculus

- Explain terminology carefully as there are many new terms in this chapter.
- Encourage learners to draw a sketch, as this is a very helpful tool in solving problems.
- It is very important that learners understand that a limit is the value that the function tends towards (gets closer and closer to) and it is not an exact answer.
- Explain to learners that the function does not have to be defined for a given value of x for the limit to exist.
- Learners must understand that the derivative of a function gives the gradient of the function at a certain point **AND** the gradient of the tangent at that point.
- Learners must practise differentiating from first principles. This often asked in examinations.
- Learners must be familiar with the different notations for differentiation.

6.1 Limits

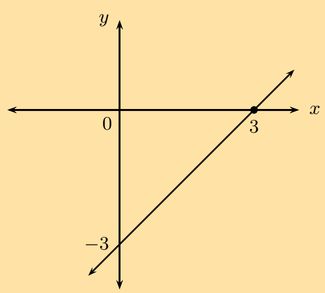
Exercise 6 – 1: Limits

1. Determine the following limits and draw a rough sketch to illustrate:

a) $\lim_{x \to 3} \frac{x^2 - 9}{x + 3}$

Solution:

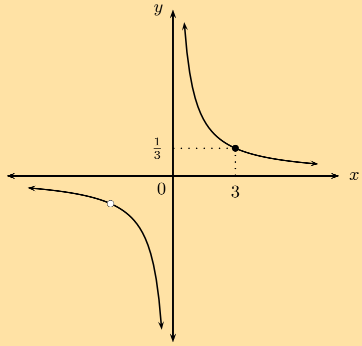
$$\begin{aligned}\lim_{x \to 3} \frac{x^2 - 9}{x + 3} &= \lim_{x \to 3} \frac{(x-3)(x+3)}{x+3} \\ &= \lim_{x \to 3} (x - 3) \\ &= 3 - 3 \\ &= 0\end{aligned}$$



b) $\lim_{x \to 3} \frac{x + 3}{x^2 + 3x}$

Solution:

$$\begin{aligned}\lim_{x \to 3} \frac{x + 3}{x^2 + 3x} &= \lim_{x \to 3} \frac{x + 3}{x(x + 3)} \\ &= \lim_{x \to 3} \frac{1}{x} \\ &= \frac{1}{3}\end{aligned}$$



2. Determine the following limits (if they exist):

a) $\lim_{x \to 2} \frac{3x^2 - 4x}{3 - x}$

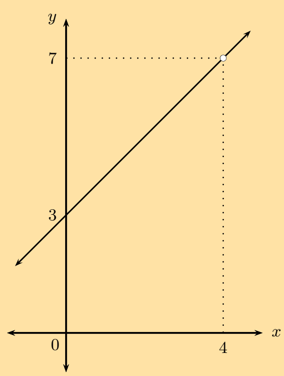
Solution:

$$\begin{aligned}\lim_{x \to 2} \frac{3x^2 - 4x}{3 - x} &= \lim_{x \to 2} \frac{3(2)^2 - 4(2)}{3 - 2} \\ &= \lim_{x \to 2} \frac{12 - 8}{1} \\ &= 4\end{aligned}$$

b) $\lim_{x \to 4} \frac{x^2 - x - 12}{x - 4}$

Solution:

$$\begin{aligned}\lim_{x \to 4} \frac{x^2 - x - 12}{x - 4} &= \lim_{x \to 4} \frac{(x - 4)(x + 3)}{x - 4} \\ &= \lim_{x \to 4} (x + 3) \\ &= 4 + 3 \\ &= 7\end{aligned}$$



Important: notice that even though the function is not defined at $x = 4$, the limit as x tends to 4 does exist and is equal to 7.

c) $\lim_{x \to 2} \left(3x + \frac{1}{3x}\right)$

Solution:

$$\begin{aligned}\lim_{x \to 2} \left(3x + \frac{1}{3x}\right) &= 6 + \frac{1}{6} \\ &= \frac{37}{6}\end{aligned}$$

d) $\lim_{x \to 0} \frac{1}{x}$

Solution:

$$\lim_{x \to 0} \frac{1}{x} = \frac{1}{0} \therefore \text{ does not exist}$$

e) $\lim_{y \to 1} \frac{y-1}{y+1}$

Solution:

$$\begin{aligned}\lim_{y \to 1} \frac{y-1}{y+1} &= \frac{1-1}{1+1} \\ &= \frac{0}{2} \\ &= 0\end{aligned}$$

f) $\lim_{y \to 1} \frac{y+1}{y-1}$

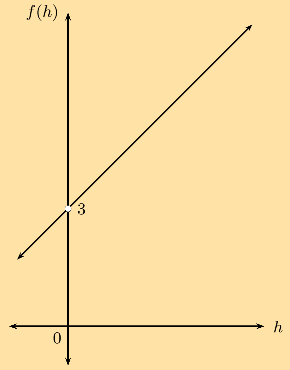
Solution:

$$\lim_{y \to 1} \frac{y+1}{y-1} = \frac{1+1}{1-1} \therefore \text{ does not exist}$$

g) $\lim_{h \to 0} \frac{3h+h^2}{h}$

Solution:

$$\begin{aligned}\lim_{h \to 0} \frac{3h+h^2}{h} &= \lim_{h \to 0} \frac{h(3+h)}{h} \\ &= \lim_{h \to 0}(3+h) \\ &= 3+0 \\ &= 3\end{aligned}$$

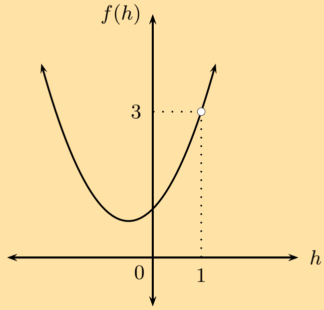


Even though the function is not defined at $h = 0$, the limit as h tends to 0 does exist and is equal to 3.

h) $\lim_{h \to 1} \frac{h^3-1}{h-1}$

Solution:

$$\begin{aligned}\lim_{h \to 1} \frac{h^3-1}{h-1} &= \lim_{h \to 1} \frac{(h-1)(h^2+h+1)}{h-1} \\ &= \lim_{h \to 1}(h^2+h+1) \\ &= 1^2+1+1 \\ &= 3\end{aligned}$$



Even though the function is not defined at $h = 1$, the limit as h tends to 1 does exist and is equal to 3.

i) $\lim_{x\to 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3}$

Solution:

$$\begin{aligned}\lim_{x\to 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3} &= \lim_{x\to 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3} \times \frac{\sqrt{x} + \sqrt{3}}{\sqrt{x} + \sqrt{3}} \\ &= \lim_{x\to 3} \frac{(x-3)}{(x-3)(\sqrt{x} + \sqrt{3})} \\ &= \frac{1}{\sqrt{x} + \sqrt{3}} \\ &= \frac{1}{\sqrt{3} + \sqrt{3}} \\ &= \frac{1}{2\sqrt{3}} \\ &= \frac{\sqrt{3}}{6}\end{aligned}$$

Note that the function is not defined at $x = 3$, but the limit as x tends to 3 does exist and is equal to $\frac{\sqrt{3}}{6}$.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28VH	1b. 28VJ	2a. 28VK	2b. 28VM	2c. 28VN	2d. 28VP
2e. 28VQ	2f. 28VR	2g. 28VS	2h. 28VT	2i. 28VV	

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Gradient at a point

Exercise 6 – 2: Gradient at a point

1. Given: $f(x) = -x^2 + 7$

   a) Find the average gradient of function f, between $x = -1$ and $x = 3$.

   Solution:

$$\begin{aligned}
\text{Average gradient} &= \frac{f(3)-f(-1)}{3-(-1)}\\
&= \frac{[-(3)^2+7]-[-(-1)^2+7]}{4}\\
&= \frac{(-9+7)-(-1+7)}{4}\\
&= \frac{-2-6}{4}\\
&= -\frac{8}{4}\\
&= -2
\end{aligned}$$

b) Illustrate this with a graph.

Solution:

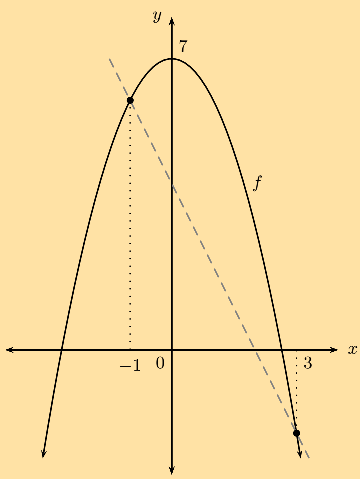
Graph:

Parabola: "frown" $(a<0)$

$y_{\text{int}}: x=0, y=7$

$x_{\text{int}}: y=0, x=\pm\sqrt{7}$

Turning point: $(0;7)$



c) Find the gradient of f at the point $x=3$ and illustrate this on your graph.

Solution:

$$\begin{aligned}
\text{Gradient at a point} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{[-(x+h)^2+7]-[-x^2+7]}{h}\\
&= \lim_{h\to 0}\frac{-x^2-2xh-h^2+7+x^2-7}{h}\\
&= \lim_{h\to 0}\frac{-2xh-h^2}{h}\\
&= \lim_{h\to 0}\frac{h(-2x-h)}{h}\\
&= \lim_{h\to 0}(-2x-h)\\
&= -2x
\end{aligned}$$

2. Determine the gradient of the tangent to g if $g(x)=\frac{3}{x}\ (x\neq 0)$ at $x=a$.

Solution:

$$\begin{aligned}
\text{Gradient of tangent} &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{\frac{3}{x+h}-\frac{3}{x}}{h}
\end{aligned}$$

To simplify the two fractions in the numerator, we get a common denominator $x(x+h)$:

$$\begin{aligned}
&= \lim_{h\to 0} \frac{\frac{3(x)-3(x+h)}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{3x-3x-3h}{h.x(x+h)} \\
&= \lim_{h\to 0} \frac{-3h}{h\ .\ x(x+h)} \\
&= \lim_{h\to 0} \frac{-3}{x^2+xh} \\
&= \frac{-3}{x^2} \\
\therefore \text{Gradient at } x = a \text{ is } &\frac{-3}{a^2}
\end{aligned}$$

3. Determine the equation of the tangent to $H(x) = x^2 + 3x$ at $x = -1$.

Solution:

$$\begin{aligned}
\text{Gradient of tangent } &= \lim_{h\to 0} \frac{H(x+h)-H(x)}{(x+h)-x} \\
&= \lim_{h\to 0} \frac{(x+h)^2+3(x+h)-(x^2+3x)}{h} \\
&= \lim_{h\to 0} \frac{x^2+2xh+h^2+3x+3h-x^2-3x}{h} \\
&= \lim_{h\to 0} \frac{2xh+3h+h^2}{h} \\
&= \lim_{h\to 0} \frac{h(2x+3+h)}{h} \\
&= 2x+3 \\
\text{At } x = -1: \quad m &= 2(-1)+3 = 1
\end{aligned}$$

The tangent meets the graph of H at $(-1;y)$

$$\therefore y = (-1)^2 + 3(-1) = -2$$

Therefore tangent goes through $H(-1;-2)$.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
\text{Substitute: } y - (-2) &= 1(x - (-1)) \\
y + 2 &= x + 1 \\
y &= x - 1 \\
\therefore \text{Eqn. tangent at } x = -1 \text{ is } y &= x - 1
\end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28VX 2. 28VY 3. 28VZ

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Exercise 6 – 3: Differentiation from first principles

1. Given: $g\left(x\right) = -x^2$

   a) Determine $\frac{g(x+h)-g(x)}{h}$.

   Solution:

$$\begin{aligned} g(x) &= -x^2 \\ g(x+h) &= -(x+h)^2 \\ \therefore \frac{g\left(x+h\right)-g\left(x\right)}{h} &= \frac{-(x+h)^2-(-x^2)}{h} \end{aligned}$$

   b) Hence, determine $\lim_{h\to 0} \frac{g\left(x+h\right)-g\left(x\right)}{h}$.

   Solution:

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{-(x+h)^2-(-x^2)}{h} \\ &= \lim_{h\to 0} \frac{-(x^2+2xh+h^2)+x^2}{h} \\ &= \lim_{h\to 0} \frac{-x^2-2xh-h^2+x^2}{h} \\ &= \lim_{h\to 0} \frac{-2xh-h^2}{h} \\ &= \lim_{h\to 0} \frac{h(-2x-h)}{h} \\ &= \lim_{h\to 0} -2x-h \\ &= -2x \end{aligned}$$

   c) Explain the meaning of your answer in (b).

   Solution:

   The derivative of $g(x)$ is $g'(x) = -2x$. The gradient of the function g is given by the expression $-2x$. The gradient of the graph depends on the value of x.

2. Find the derivative of $f\left(x\right) = -2x^2 + 3x + 1$ using first principles.

   Solution:

$$\begin{aligned} f(x) &= -2x^2+3x+1 \\ f(x+h) &= -2(x+h)^2+3(x+h)+1 \\ f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{-2(x+h)^2+3(x+h)+1-(-2x^2+3x+1)}{h} \\ &= \lim_{h\to 0} \frac{-2(x^2+2xh+h^2)+1+3x+3h+2x^2-3x-1}{h} \\ &= \lim_{h\to 0} \frac{-2x^2-4xh-2h^2+3h+2x^2}{h} \\ &= \lim_{h\to 0} \frac{-4xh-2h^2+3h}{h} \\ &= \lim_{h\to 0} \frac{h(-4x-2h+3)}{h} \\ &= \lim_{h\to 0} (-4x-2h+3) \\ f'(x) &= -4x+3 \end{aligned}$$

3. Determine the derivative of $f(x) = \frac{1}{x-2}$ using first principles.

Solution:

$$\begin{aligned}
f(x) &= \frac{1}{x-2} \\
f(x+h) &= \frac{1}{x+h-2} \\
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+h-2} - \frac{1}{x-2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{(x-2)-(x+h-2)}{(x+h-2)(x-2)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x-2-x-h+2}{(x+h-2)(x-2)}}{h} \\
&= \lim_{h\to 0} \left(\frac{-h}{(x+h-2)(x-2)}\right) \times \frac{1}{h} \\
&= \lim_{h\to 0} \frac{-1}{(x+h-2)(x-2)} \\
f'(x) &= \frac{-1}{(x-2)^2}
\end{aligned}$$

4. Determine $g'(3)$ from first principles if $g(x) = -5x^2$.

Solution:

$$\begin{aligned}
g(x) &= -5x^2 \\
g(x+h) &= -5(x+h)^2 \\
g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{-5(x^2+2xh+h^2)-(-5x^2)}{h} \\
&= \lim_{h\to 0} \frac{-5x^2-10xh-5h^2+5x^2}{h} \\
&= \lim_{h\to 0} \frac{-10xh-5h^2}{h} \\
&= \lim_{h\to 0} \frac{h(-10x-5h)}{h} \\
&= \lim_{h\to 0} (-10x-5h) \\
&= -10x
\end{aligned}$$

Therefore:

$$\begin{aligned}
g'(3) &= -10(3) \\
&= -30
\end{aligned}$$

5. If $p(x) = 4x(x-1)$, determine $p'(x)$ using first principles.

Solution:

$$\begin{aligned} p(x) &= 4x(x-1) \\ &= 4x^2 - 4x \\ p(x+h) &= 4(x+h)^2 - 4(x+h) \\ p'(x) &= \lim_{h\to 0} \frac{p(x+h)-p(x)}{h} \\ &= \lim_{h\to 0} \frac{4(x^2+2xh+h^2)-4x-4h-4x^2+4x}{h} \\ &= \lim_{h\to 0} \frac{4x^2+8xh+4h^2-4h-4x^2}{h} \\ &= \lim_{h\to 0} \frac{h(8x+4h-4)}{h} \\ &= \lim_{h\to 0}(8x+4h-4) \\ &= 8x-4 \end{aligned}$$

6. Find the derivative of $k(x) = 10x^3$ using first principles.

   Solution:

$$\begin{aligned} k(x) &= 10x^3 \\ k(x+h) &= 10(x+h)^3 \\ k'(x) &= \lim_{h\to 0} \frac{k(x+h)-k(x)}{h} \\ &= \lim_{h\to 0} \frac{10(x+h)^3-10x^3}{h} \\ &= \lim_{h\to 0} \frac{10\left(x^3+3x^2h+3xh^2+h^3\right)-10x^3}{h} \\ &= \lim_{h\to 0} \frac{10x^3+30x^2h+30xh^2+10h^3-10x^3}{h} \\ &= \lim_{h\to 0} \frac{30x^2h+30xh^2+10h^3}{h} \\ &= \lim_{h\to 0}\left(30x^2+30xh+10h^2\right) \\ k'(x) &= 30x^2 \end{aligned}$$

7. Differentiate $f(x) = x^n$ using first principles.
   (Hint: Use Pascal's triangle)

   Solution:

   Step 1: Use first principles to find the derivative.

$$f'(x) = \lim_{h\to 0} \text{(average gradient)}$$

$$\begin{aligned} \text{Average gradient} &= \frac{f(x+h)-f(x)}{h} \\ &= \frac{(x+h)^n - x^n}{h} \end{aligned}$$

   Expand $(x+h)^n$ using the pattern of the coefficients given by Pascal's Triangle:

$$\begin{array}{lccccccccc} x & & & & & 1 & & & & \\ (x+h) & & & & 1 & & 1 & & & \\ (x+h)^2 & & & 1 & & 2 & & 1 & & \\ (x+h)^3 & & 1 & & 3 & & 3 & & 1 & \\ (x+h)^4 & 1 & & 4 & & 6 & & 4 & & 1 \\ . & & & & & & & & & \\ . & & & & & & & & & \\ (x+h)^n & 1 & n & . & . & . & . & . & n & 1 \end{array}$$

$$\therefore (x+h)^n = x^n + nx^{n-1}h \cdots nxh^{n-1} + h^n$$

All the terms, except x^n, contain h.

$$\begin{aligned}
\text{Average gradient } &= \frac{(x^n + nx^{n-1}h + \cdots + nxh^{n-1} + h^n) - x^n}{h} \\
&= \frac{nx^{n-1}h + \cdots + nxh^{n-1} + h^n}{h} \\
&= \frac{h(nx^{n-1} + \cdots + nxh^{n-2} + h^{n-1})}{h} \\
&= nx^{n-1} + \cdots + nxh^{n-2} + h^{n-1}
\end{aligned}$$

$$\begin{aligned}
\therefore \text{ Average gradient } &= \lim_{h \to 0} \left(nx^{n-1} + \cdots + nxh^{n-2} + h^{n-1}\right) \\
\therefore &= nx^{n-1}
\end{aligned}$$

$$\begin{aligned}
\text{If } f(x) &= x^n \\
f'(x) &= nx^{n-1}
\end{aligned}$$

This is a very valuable general rule for finding the derivative of a function.

Check answers online with the exercise code below or click on 'show me the answer'.
1a. 28W3 1b. 28W4 1c. 28W5 2. 28W6 3. 28W7 4. 28W8
5. 28W9 6. 28WB 7. 28WC

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6.3 Rules for differentiation

Exercise 6 – 4: Rules for differentiation

1. Differentiate the following:

   a) $y = 3x^2$
      Solution:
      $\frac{dy}{dx} = 6x$

   b) $f(x) = 25x$
      Solution:
      $f'(x) = 25$

   c) $k(x) = -30$
      Solution:
      $k'(x) = 0$

   d) $y = -4x^5 + 2$
      Solution:
      $\frac{dy}{dx} = -20x^4$

   e) $g(x) = 16x^{-2}$
      Solution:
      $g'(x) = -32x^{-3} = -\frac{32}{x^3}$

   f) $y = 10(7 - 3)$
      Solution:
      $y = 40 \therefore y' = 0$

g) $q(x) = x^4 - 6x^2 - 1$
Solution:
$q'(x) = 4x^3 - 12x$

h) $y = x^2 + x + 4$
Solution:
$\frac{dy}{dx} = 2x + 1$

i) $f(x) = \frac{1}{3}x^3 - x^2 + \frac{2}{5}$
Solution:
$f'(x) = x^2 - 2x$

j) $y = 3x^{\frac{3}{2}} - 4x + 20$
Solution:
$\frac{dy}{dx} = \frac{9}{2}x^{\frac{1}{2}} - 4$

k) $g(x) = x(x+2) + 5x$
Solution:
$g(x) = x^2 + 7x$
$g'(x) = 2x + 7$

l) $p(x) = 200[x^3 - \frac{1}{2}x^2 + \frac{1}{5}x - 40]$
Solution:
$p'(x) = 200[3x^2 - x + \frac{1}{5}]$
$p'(x) = 600x^2 - 200x + 40$

m) $y = 14(x-1)\left[\frac{1}{2} + x^2\right]$
Solution:
$y = 14(x^3 - x^2 + \frac{1}{2}x - \frac{1}{2})$
$\therefore \frac{dy}{dx} = 14\left(3x^2 - 2x + \frac{1}{2}\right)$
$= 42x^2 - 28x + 7$

2. Find $f'(x)$ if $f(x) = \frac{x^2-5x+6}{x-2}$.

Solution:

$$\begin{aligned} f(x) &= \frac{x^2 - 5x + 6}{x - 2} \\ &= \frac{(x-3)(x-2)}{x-2} \\ &= x - 3 \\ f'(x) &= 1 \end{aligned}$$

3. Find $f'(y)$ if $f(y) = \sqrt{y}$.

Solution:

$$\begin{aligned} f(y) &= \sqrt{y} \\ &= y^{\frac{1}{2}} \\ f'(y) &= \frac{1}{2}y^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{y}} \end{aligned}$$

4. Find $f'(z)$ if $f(z) = (z-1)(z+1)$.

Solution:

$$\begin{aligned} f(z) &= (z-1)(z+1) \\ &= z^2 - 1 \\ f'(z) &= 2z \end{aligned}$$

5. Determine $\frac{dy}{dx}$ if $y = \frac{x^3+2\sqrt{x}-3}{x}$.

Solution:

$$\begin{aligned} y &= \frac{x^3 + 2\sqrt{x} - 3}{x} \\ y &= x^2 + 2x^{-\frac{1}{2}} - 3x^{-1} \\ \frac{dy}{dx} &= 2x + 2\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} - 3(-1)x^{-2} \\ &= 2x - \frac{1}{\sqrt{x^3}} + \frac{3}{x^2} \end{aligned}$$

6. Determine the derivative of $y = \sqrt{x^3} + \frac{1}{3x^3}$.

Solution:

$$\begin{aligned} y &= x^{\frac{3}{2}} + \frac{1}{3}x^{-3} \\ \frac{dy}{dx} &= \frac{3}{2}x^{\frac{1}{2}} - x^{-4} \\ &= \frac{3}{2}\sqrt{x} - \frac{1}{x^4} \end{aligned}$$

7. Find $D_x\left[x^{\frac{3}{2}} - \frac{3}{x^{\frac{1}{2}}}\right]^2$.

Solution:

$$\begin{aligned} D_x\left[x^{\frac{3}{2}} - \frac{3}{x^{\frac{1}{2}}}\right]^2 &= D_x\left[x^{\frac{3}{2}} - \frac{3}{x^{\frac{1}{2}}}\right]\left[x^{\frac{3}{2}} - \frac{3}{x^{\frac{1}{2}}}\right] \\ &= D_x\left[x^3 - 2(3x) + \frac{9}{x}\right] \\ &= D_x\left[x^3 - 6x + 9x^{-1}\right] \\ &= 3x^2 - 6 - 9x^{-2} \\ &= 3x^2 - 6 - \frac{9}{x^2} \end{aligned}$$

8. Find $\frac{dy}{dx}$ if $x = 2y + 3$.

Solution:

Make y the subject of the formula in order to differentiate y with respect to x.

$$\begin{aligned} y &= \frac{1}{2}x - \frac{3}{2} \\ \therefore \frac{dy}{dx} &= \frac{1}{2} \end{aligned}$$

9. Determine $f'(\theta)$ if $f(\theta) = 2(\theta^{\frac{3}{2}} - 3\theta^{-\frac{1}{2}})^2$.

Solution:

$$\begin{aligned} f(\theta) &= 2(\theta^{\frac{3}{2}} - 3\theta^{-\frac{1}{2}})^2 \\ &= 2(\theta^{\frac{3}{2}} - 3\theta^{-\frac{1}{2}})(\theta^{\frac{3}{2}} - 3\theta^{-\frac{1}{2}}) \\ &= 2(\theta^3 - 6\theta + 9\theta^{-1}) \\ \therefore f'(\theta) &= 2(3\theta^2 - 6 - 9\theta^{-2}) \\ &= 2(3\theta^2 - 6 - \frac{9}{\theta^2}) \\ &= 6\theta^2 - 12 - \frac{18}{\theta^2} \end{aligned}$$

10. Find $\frac{dp}{dt}$ if $p(t) = \frac{(t+1)^3}{\sqrt{t}}$.

Solution:

$$\begin{aligned}
p(t) &= \frac{(t+1)^3}{\sqrt{t}} \\
&= \frac{t^3 + 3t^2 + 3t + 1}{t^{\frac{1}{2}}} \\
&= t^{-\frac{1}{2}}(t^3 + 3t^2 + 3t + 1) \\
&= t^{\frac{5}{2}} + 3t^{\frac{3}{2}} + 3t^{\frac{1}{2}} + t^{-\frac{1}{2}} \\
\therefore \frac{dp}{dt} &= \frac{5}{2}t^{\frac{3}{2}} + \frac{9}{2}t^{\frac{1}{2}} + \frac{3}{2}t^{-\frac{1}{2}} - \frac{1}{2}t^{-\frac{3}{2}} \\
&= \frac{5}{2}t^{\frac{3}{2}} + \frac{9}{2}t^{\frac{1}{2}} + \frac{3}{2t^{\frac{1}{2}}} - \frac{1}{2t^{\frac{3}{2}}}
\end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28WF	1b. 28WG	1c. 28WH	1d. 28WJ	1e. 28WK	1f. 28WM
1g. 28WN	1h. 28WP	1i. 28WQ	1j. 28WR	1k. 28WS	1l. 28WT
1m. 28WV	2. 28WW	3. 28WX	4. 28WY	5. 28WZ	6. 28X2
7. 28X3	8. 28X4	9. 28X5	10. 28X6		

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6.4 Equation of a tangent to a curve

Exercise 6 – 5: Equation of a tangent to a curve

1. Determine the equation of the tangent to the curve defined by $F(x) = x^3 + 2x^2 - 7x + 1$ at $x = 2$.

Solution:

$$\begin{aligned}
\text{Gradient of tangent } &= F'(x) \\
F'(x) &= 3x^2 + 4x - 7 \\
F'(2) &= 3(2)^2 + (4)(2) - 7 \\
&= 13 \\
\therefore \text{ Tangent: } y &= 13x + c
\end{aligned}$$

where c is the y-intercept.
Tangent meets $F(x)$ at $(2; F(2))$

$$\begin{aligned}
F(2) &= (2)^3 + 2(2)^2 - 7(2) + 1 \\
&= 8 + 8 - 14 + 1 \\
&= 3 \\
\text{Tangent: } 3 &= 13(2) + c \\
\therefore c &= -23 \\
y &= 13x - 23
\end{aligned}$$

2. Determine the point where the gradient of the tangent to the curve:

a) $f(x) = 1 - 3x^2$ is equal to 5.
Solution:

$$\begin{aligned}
\text{Gradient of tangent } &= f'(x) = -6x \\
\therefore -6x &= 5 \\
\therefore x &= -\frac{5}{6} \\
\text{And } f\left(-\frac{5}{6}\right) &= 1 - 3\left(-\frac{5}{6}\right)^2 \\
&= 1 - 3\left(\frac{25}{36}\right) \\
&= 1 - \frac{25}{12} \\
&= -\frac{13}{12} \\
&\therefore \left(-\frac{5}{6}; -\frac{13}{12}\right)
\end{aligned}$$

b) $g(x) = \frac{1}{3}x^2 + 2x + 1$ is equal to 0.
Solution:

$$\begin{aligned}
\text{Gradient of tangent } &= g'(x) = \frac{2}{3}x + 2 \\
\therefore \frac{2}{3}x + 2 &= 0 \\
\frac{2}{3}x &= -2 \\
\therefore x &= -2 \times \frac{3}{2} \\
&= -3 \\
\text{And } g(-3) &= \frac{1}{3}(-3)^2 + 2(-3) + 1 \\
&= \frac{1}{3}(9) - 6 + 1 \\
&= 3 - 6 + 1 \\
&= -2 \\
&\therefore (-3; -2)
\end{aligned}$$

3. Determine the point(s) on the curve $f(x) = (2x - 1)^2$ where the tangent is:

a) parallel to the line $y = 4x - 2$.
Solution:

$$\begin{aligned}
\text{Gradient of tangent } &= f'(x) \\
f(x) &= (2x - 1)^2 \\
&= 4x^2 - 4x + 1 \\
\therefore f'(x) &= 8x - 4 \\
\text{Tangent is parallel to } y &= 4x - 2 \\
\therefore m &= 4 \\
\therefore f'(x) = 8x - 4 &= 4 \\
8x &= 8 \\
x &= 1 \\
\text{For } x = 1: \quad y &= (2(1) - 1)^2 \\
&= 1
\end{aligned}$$

Therefore, the tangent is parallel to the given line at the point $(1; 1)$.

b) perpendicular to the line $2y + x - 4 = 0$.

Solution:

$$\text{Perpendicular to } 2y + x - 4 = 0$$
$$y = -\frac{1}{2}x + 2$$
$$\therefore \text{ gradient of } \perp \text{ line } = 2 \quad (m_1 \times m_2 = -1)$$
$$\therefore f'(x) = 8x - 4$$
$$\therefore 8x - 4 = 2$$
$$8x = 6$$
$$x = \frac{3}{4}$$
$$\therefore y = \left[2\left(\frac{3}{4}\right) - 1\right]^2$$
$$= \frac{1}{4}$$
$$\therefore \left(\frac{3}{4}; \frac{1}{4}\right)$$

Therefore, the tangent is perpendicular to the given line at the point $\left(\frac{3}{4}; \frac{1}{4}\right)$.

4. Given the function f: $y = -x^2 + 4x - 3$.

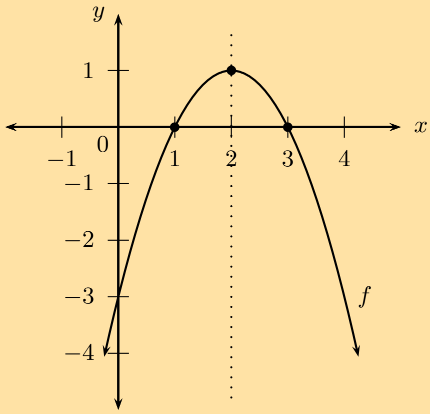
a) Draw a graph of f, indicating all intercepts and turning points.

Solution:

Complete the square:

$$y = -[x^2 - 4x + 3]$$
$$= -[(x-2)^2 - 4 + 3]$$
$$= -(x-2)^2 + 1$$
$$\text{Turning point}: (2; 1)$$

Intercepts:
$y_{\text{int}}: x = 0, y = -3$
$x_{\text{int}}: y = 0,$
$-x^2 + 4x - 3 = 0$
$x^2 - 4x + 3 = 0$
$(x-3)(x-1) = 0$
$x = 3$ or $x = 1$
Shape: "frown" $(a < 0)$



b) Find the equations of the tangents to f at:

i. the y-intercept of f.
ii. the turning point of f.

iii. the point where $x = 4{,}25$.

Solution:

i.

$$\begin{aligned} y_{\text{int}} &: (0;-3) \\ m_{\text{tangent}} &= f'(x) = -2x + 4 \\ f'(0) &= -2(0) + 4 \\ \therefore m &= 4 \\ \text{Tangent } y &= 4x + c \\ \text{Through } (0;-3) &\therefore y = 4x - 3 \end{aligned}$$

ii.

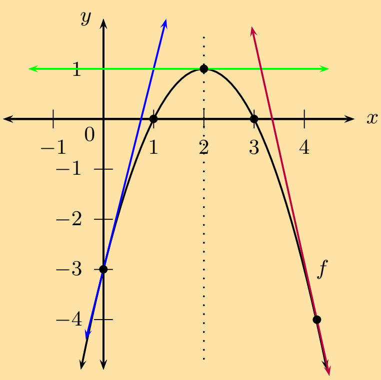
$$\begin{aligned} \text{Turning point: } &(2;1) \\ m_{\text{tangent}} &= f'(2) = -2(2) + 4 \\ &= 0 \\ \text{Tangent equation } &y = 1 \end{aligned}$$

iii.

$$\begin{aligned} \text{If } x &= 4{,}25 \\ f(4{,}25) &= -4{,}25^2 + 4(4{,}25) - 3 \\ &= -4{,}0625 \\ m_{\text{tangent}} \text{ at } x &= 4{,}25 \\ m &= -2(4{,}25) + 4 \\ &= -4{,}5 \\ \text{Tangent } y &= -4{,}5x + c \\ \text{Through } (4{,}25;-4{,}0625) & \\ -4{,}0625 &= -4{,}5(4{,}25) + c \\ \therefore c &= 15{,}0625 \\ y &= -4{,}5x + 15{,}0625 \end{aligned}$$

c) Draw the three tangents above on your graph of f.

Solution:



d) Write down all observations about the three tangents to f.

Solution:

Tangent at y_{int} (blue line): gradient is positive, the function is increasing at this point.
Tangent at turning point (green line): gradient is zero, tangent is a horizontal line, parallel to x-axis.
Tangent at $x = 4{,}25$ (purple line): gradient is negative, the function is decreasing at this point.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28X7 2a. 28X8 2b. 28X9 3. 28XB 4. 28XC

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Exercise 6 – 6: Second derivative

1. Calculate the second derivative for each of the following:

   a) $g(x) = 5x^2$

   Solution:

$$g'(x) = 10x$$
$$g''(x) = 10$$

   b) $y = 8x^3 - 7x$

   Solution:

$$\frac{dy}{dx} = 24x^2 - 7$$
$$\frac{d^2y}{dx^2} = 48x$$

   c) $f(x) = x(x-6) + 10$

   Solution:

$$f(x) = x^2 - 6x + 10$$
$$f'(x) = 2x - 6$$
$$f''(x) = 2$$

   d) $y = x^5 - x^3 + x - 1$

   Solution:

$$\frac{dy}{dx} = 5x^4 - 3x^2 + 1$$
$$\frac{d^2y}{dx^2} = 20x^3 - 6x$$

   e) $k(x) = (x^2+1)(x-1)$

   Solution:

$$k(x) = x^3 - x^2 + x - 1$$
$$k'(x) = 3x^2 - 2x + 1$$
$$k''(x) = 6x - 2$$

   f) $p(x) = -\frac{10}{x^2}$

   Solution:

$$p(x) = -10x^{-2}$$
$$p'(x) = 20x^{-3} = \frac{20}{x^3}$$
$$p''(x) = -60x^{-4} = \frac{-60}{x^4}$$

   g) $q(x) = \sqrt{x} + 5x^2$

Solution:

$$\begin{aligned} q(x) &= x^{\frac{1}{2}} + 5x^2 \\ q'(x) &= \frac{1}{2}x^{-\frac{1}{2}} + 10x \\ q''(x) &= -\frac{1}{4}x^{-\frac{3}{2}} + 10 \\ &= -\frac{1}{4\sqrt{x^3}} + 10 \end{aligned}$$

2. Find the first and second derivatives of $f(x) = 5x(2x+3)$.

   Solution:

$$\begin{aligned} f(x) &= 10x^2 + 15x \\ f'(x) &= 20x + 15 \\ f''(x) &= 20 \end{aligned}$$

3. Find $\frac{d^2}{dx^2}\left[6\sqrt[3]{x^2}\right]$.

   Solution:

$$\begin{aligned} y &= 6\sqrt[3]{x^2} \\ &= 6x^{\frac{2}{3}} \\ \frac{dy}{dx} &= 6\left(\frac{2}{3}\right)x^{\frac{2}{3}-1} \\ &= 4x^{-\frac{1}{3}} \\ \frac{d^2y}{dx^2} &= 4\left(-\frac{1}{3}\right)x^{-\frac{1}{3}-1} \\ &= -\frac{4}{3}x^{-\frac{4}{3}} \\ &= -\frac{4}{3\sqrt[3]{x^4}} \end{aligned}$$

4. Given the function $g: \quad y = (1-2x)^3$.

   a) Determine g' and g''.

   Solution:

$$\begin{aligned} g(x) &= (1-2x)^3 \\ &= (1-2x)(1-2x)^2 \\ &= (1-2x)(1-4x+4x^2) \\ &= 1 - 6x + 12x^2 - 8x^3 \\ g'(x) &= -6 + 24x - 24x^2 \\ g''(x) &= 24 - 48x \end{aligned}$$

   b) What type of function is:

      i. g'
      ii. g''

   Solution:

      i. $g'(x) = -6 + 24x - 24x^2$ (quadratic function)

      ii. $g''(x) = 24 - 48x$ (linear function)

c) Find the value of $g''\left(\frac{1}{2}\right)$.

Solution:

$$\begin{aligned} g''\left(\frac{1}{2}\right) &= 24 - 48\left(\frac{1}{2}\right) \\ &= 24 - 24 \\ &= 0 \end{aligned}$$

d) What do you observe about the degree (highest power) of each of the derived functions?

Solution:

Each derivative function is one degree lower than the previous one.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28XD 1b. 28XF 1c. 28XG 1d. 28XH 1e. 28XJ 1f. 28XK
1g. 28XM 2. 28XN 3. 28XP 4. 28XQ

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6.6 Sketching graphs

Functions of the form $y = ax^3 + bx^2 + cx + d$

Intercepts

Exercise 6 – 7: Intercepts

1. Given the function $f(x) = x^3 + x^2 - 10x + 8$.

a) Determine the x- and y-intercepts of $f(x)$.

Solution:

For the y-intercept, let $x = 0$:

$$\begin{aligned} f(0) &= (0)^3 + (0)^2 - 10(0) + 8 \\ &= 8 \end{aligned}$$

This gives the point $(0; 8)$.

We use the factor theorem to find a factor of $f(x)$ by trial and error:

$$\begin{aligned} f(x) &= x^3 + x^2 - 10x + 8 \\ f(1) &= (1)^3 + (1)^2 - 10(1) + 8 \\ &= 0 \\ \therefore (x-1) &\text{ is a factor of } f(x) \end{aligned}$$

Factorise further by inspection:

$$\begin{aligned} f(x) &= (x-1)(x^2 + 2x - 8) \\ &= (x-1)(x-2)(x+4) \end{aligned}$$

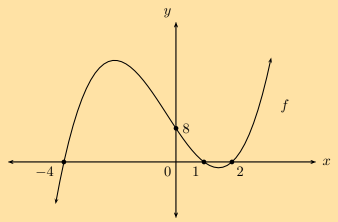
For the x-intercept, let $f(x) = 0$:

$$0 = (x-1)(x-2)(x+4)$$
$$\therefore x = 1, x = 2 \text{ or } x = -4$$

This gives the points $(-4;0)$, $(1;0)$ and $(2;0)$.

b) Draw a rough sketch of the graph.

Solution:



c) Is the function increasing or decreasing at $x = -5$?

Solution:

Increasing

2. Determine the x- and y-intercepts for each of the following:

a) $y = -x^3 - 5x^2 + 9x + 45$

Solution:

For the y-intercept, let $x = 0$:

$$\begin{aligned} y &= -(0)^3 - 5(0)^2 + 9(0) + 45 \\ &= 45 \end{aligned}$$

This gives the point $(0;45)$.

We use the factor theorem to find a factor by trial and error:

$$\begin{aligned} \text{Let } f(x) &= -x^3 - 5x^2 + 9x + 45 \\ f(3) &= -(3)^3 - 5(3)^2 + 9(3) + 45 \\ &= 0 \\ \therefore (x-3) &\text{ is a factor of } f(x) \end{aligned}$$

Factorise further by inspection:

$$\begin{aligned} f(x) &= (x-3)(-x^2 - 8x - 15) \\ &= -(x-3)(x^2 + 8x + 15) \\ &= -(x-3)(x+3)(x+5) \end{aligned}$$

For the x-intercept, let $y = 0$:

$$0 = -(x-3)(x+3)(x+5)$$
$$\therefore x = 3, x = -3 \text{ or } x = -5$$

This gives the points $(-5;0)$, $(-3;0)$ and $(3;0)$.

b) $y = x^3 - \frac{5}{4}x^2 - \frac{7}{4}x + \frac{1}{2}$

Solution:

For the y-intercept, let $x = 0$:

$$\begin{aligned} y &= (0)^3 - \frac{5}{4}(0)^2 - \frac{7}{4}(0) + \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

This gives the point $(0;\frac{1}{2})$.
We use the factor theorem to find a factor by trial and error:

$$\begin{aligned}
\text{Let } f(x) &= x^3 - \frac{5}{4}x^2 - \frac{7}{4}x + \frac{1}{2} \\
f(-1) &= (-1)^3 - \frac{5}{4}(-1)^2 - \frac{7}{4}(-1) + \frac{1}{2} \\
&= -1 - \frac{5}{4} + \frac{7}{4} + \frac{1}{2} \\
&= -\frac{4}{4} - \frac{5}{4} + \frac{7}{4} + \frac{2}{4} \\
&= 0 \\
\therefore (x+1) &\text{ is a factor of } f(x)
\end{aligned}$$

Factorise further by inspection:

$$\begin{aligned}
f(x) &= (x+1)(x^2 - \frac{9}{4}x + \frac{1}{2}) \\
&= (x+1)(x-2)\left(x - \frac{1}{4}\right)
\end{aligned}$$

For the x-intercept, let $y = 0$:

$$\begin{aligned}
0 &= (x+1)(x-2)\left(x - \frac{1}{4}\right) \\
\therefore x = -1, x = 2 &\text{ or } x = \frac{1}{4}
\end{aligned}$$

This gives the points $(-1;0)$, $\left(\frac{1}{4};0\right)$, $(2;0)$ and $\left(0;\frac{1}{2}\right)$.

c) $y = x^3 - x^2 - 12x + 12$

Solution:

For the y-intercept, let $x = 0$:

$$\begin{aligned}
y &= (0)^3 - (0)^2 - 12(0) + 12 \\
&= 12
\end{aligned}$$

This gives the point $(0;12)$.
We use the factor theorem to find a factor by trial and error:

$$\begin{aligned}
\text{Let } f(x) &= x^3 - x^2 - 12x + 12 \\
f(1) &= (1)^3 - (1)^2 - 12(1) + 12 \\
&= 0 \\
\therefore (x-1) &\text{ is a factor of } f(x)
\end{aligned}$$

Factorise further by inspection:

$$\begin{aligned}
f(x) &= (x-1)(x^2 - 12) \\
&= (x-1)(x - \sqrt{12})(x + \sqrt{12})
\end{aligned}$$

For the x-intercept, let $y = 0$:

$$\begin{aligned}
0 &= (x-1)(x - \sqrt{12})(x + \sqrt{12}) \\
\therefore x = 1, x = \sqrt{12} &\text{ or } x = -\sqrt{12}
\end{aligned}$$

This gives the points $(1;0)$, $(\sqrt{12};0)$ and $(-\sqrt{12};0)$.

d) $y = x^3 - 16x$

Solution:

For the y-intercept, let $x = 0$:

$$\begin{aligned} y &= (0)^3 - 16(0) \\ &= 0 \end{aligned}$$

This gives the point $(0; 0)$.

We take out a common factor of x and then factorise the difference of two squares:

$$\begin{aligned} y &= x(x^2 - 16) \\ &= x(x-4)(x+4) \end{aligned}$$

For the x-intercept, let $y = 0$:

$$\begin{aligned} 0 &= x(x-4)(x+4) \\ \therefore x &= 0, x = 4 \text{ or } x = -4 \end{aligned}$$

This gives the points $(0; 0)$, $(-4; 0)$ and $(4; 0)$.

e) $y = x^3 - 5x^2 + 6$

Solution:

For the y-intercept, let $x = 0$:

$$\begin{aligned} y &= (0)^3 - 5(0)^2 + 6 \\ &= 6 \end{aligned}$$

This gives the point $(0; 6)$.

We use the factor theorem to find a factor by trial and error:

$$\begin{aligned} f(x) &= x^3 - 5x^2 + 6 \\ f(-1) &= (-1)^3 - 5(-1)^2 + 6 \\ &= -1 - 5 + 6 \\ &= 0 \\ \therefore (x+1) &\text{ is a factor of } f(x) \end{aligned}$$

Factorise further by inspection:

$$f(x) = (x+1)(x^2 - 6x + 6)$$

For the x-intercept, let $y = 0$:

$$\begin{aligned} 0 &= (x+1)(x^2 - 6x + 6) \\ \therefore x = -1, \text{ or } x &= \frac{-(-6) \pm \sqrt{6^2 - 4(1)(6)}}{2(1)} \\ &= \frac{6 \pm \sqrt{36 - 24}}{2} \\ &= \frac{6 \pm \sqrt{12}}{2} \\ &= \frac{6 \pm 2\sqrt{3}}{2} \\ &= 3 \pm \sqrt{3} \end{aligned}$$

This gives the points $(-1; 0)$, $(3 - \sqrt{3}; 0)$ and $(3 + \sqrt{3}; 0)$.

3. Determine all intercepts for $g(x) = x^3 + 3x^2 - 10x$ and draw a rough sketch of the graph.

Solution:

For the y-intercept, let $x = 0$:

$$\begin{aligned} y &= (0)^3 + 3(0)^2 - 10(0) \\ &= 0 \end{aligned}$$

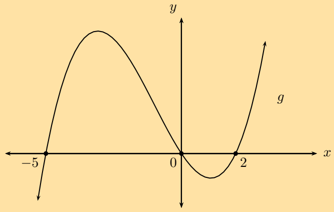
This gives the point $(0; 0)$.

$$\begin{aligned} y &= x^3 + 3x^2 - 10x \\ &= x(x^2 + 3x - 10) \\ &= x(x+5)(x-2) \end{aligned}$$

For the x-intercept, let $y = 0$:

$$\begin{aligned} 0 &= x(x+5)(x-2) \\ \therefore x &= 0, x = -5 \text{ or } x = 2 \end{aligned}$$

This gives the points $(0; 0)$, $(-5; 0)$ and $(2; 0)$.



Check answers online with the exercise code below or click on 'show me the answer'.
1. 28XR 2a. 28XS 2b. 28XT 2c. 28XV 2d. 28XW 2e. 28XX
3. 28XY

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Stationary points

Exercise 6 – 8: Stationary points

1. Use differentiation to determine the stationary point(s) for $g(x) = -x^2 + 5x - 6$.

Solution:

$$g(x) = -x^2 + 5x - 6$$
$$g'(x) = -2x + 5$$
At stationary point: $y' = 0$
$$-2x + 5 = 0$$
$$-2x = -5$$
$$x = 2{,}5$$
Substitute $x = 2{,}5$
$$\therefore y = -(2{,}5)^2 + 5(2{,}5) - 6$$
$$= -6{,}25 + 12{,}5 - 6$$
$$= 0{,}25$$
$\therefore$ Stationary point $\left(\frac{5}{2}; \frac{1}{4}\right)$

2. Determine the x-values of the stationary points for $f(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x + 5$.

   Solution:

$$f(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x + 5$$
$$f'(x) = -x^2 + x + 6$$
At stationary point: $y' = 0$
$$-x^2 + x + 6 = 0$$
$$x^2 - x - 6 = 0$$
$$(x + 2)(x - 3) = 0$$
$\therefore x = -2$ or $x = 3$

3. Find the coordinates of the stationary points of the following functions using the rules of differentiation:

   a) $y = (x - 1)^3$

   Solution:

$$y = (x - 1)^3$$
$$y = x^3 - 3x^2 + 3x - 1$$
$$y' = 3x^2 - 6x + 3$$
At stationary point: $y' = 0$
$$3(x^2 - 2x + 1) = 0$$
$$(x - 1)^2 = 0$$
$$\therefore x = 1$$
Substitute $x = 1$
$$y = (1 - 1)^3 = 0$$
$\therefore$ Stationary point $(1; 0)$

   b) $y = x^3 - 5x^2 + 1$

   Solution:

$$y = x^3 - 5x^2 + 1$$
$$y' = 3x^2 - 10x$$
At stationary point: $y' = 0$
$$3x^2 - 10x = 0$$
$$x(3x - 10) = 0$$
$$\therefore x = 0, x = \frac{10}{3}$$

$$\text{Substitute } x = 0$$
$$y = (0)^3 - 5(0)^2 + 1 = 1$$
$$\therefore \text{ Stationary point } (0; 1)$$
$$\text{Substitute } x = \frac{10}{3}$$
$$y = \left(\frac{10}{3}\right)^3 - 5\left(\frac{10}{3}\right)^2 + 1$$
$$= \frac{1000}{27} - \frac{500}{9} + 1$$
$$= -\frac{473}{27}$$
$$\therefore \text{ Stationary point } \left(\frac{10}{3}; -\frac{473}{27}\right)$$

c) $y + 7x = 1$

Solution:

$$y = -7x + 1$$
$$y' = -7$$

This is a straight line with a constant gradient, therefore there is no stationary point.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28XZ 2. 28Y2 3a. 28Y3 3b. 28Y4 3c. 28Y5

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Sketching cubic graphs

Exercise 6 – 9: Concavity and points of inflection

Complete the following for each function:

- Determine and discuss the change in gradient of the function.
- Determine the concavity of the graph.
- Find the inflection point.
- Draw a sketch of the graph.

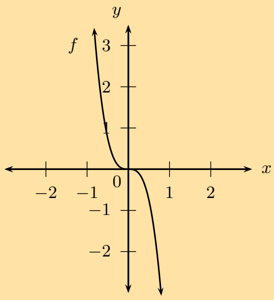
1. $f : y = -2x^3$

   Solution:

   a) Gradient: $f'(x) = -6x^2$
   $f'(x) < 0$ for $x < 0$: function decreasing
   $f'(x) = 0$ for $x = 0$: function stationary
   $f'(x) > 0$ for $x > 0$: function increasing

   b) Concavity: $f''(x) = -12x$
   $f''(x) > 0$ for $x < 0$: concave up
   $f''(x) = 0$ for $x = 0$: point of inflection
   $f''(x) < 0$ for $x > 0$: concave down

   c) Point of inflection: $(0; 0)$



2. $g(x) = \frac{1}{8}x^3 + 1$

   Solution:

   $g(x) = y = \frac{1}{8}x^3 + 1$

   a) Gradient: $g'(x) = \frac{3}{8}x^2$
   Gradient always > 0 (since $\frac{3}{8}x^2 \geq 0$) for all x except $x = 0$ where the gradient will be 0.

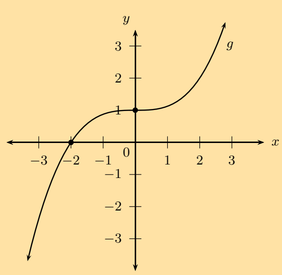
   b) Concavity: $g''(x) = \frac{3}{4}x$
   $g''(x) < 0$ for $x < 0$: concave down
   $g''(x) = 0$ for $x = 0$: point of inflection
   $g''(x) > 0$ for $x > 0$: concave up

   c) Point of inflection: $(0; 1)$

   Intercepts:

$$
\begin{aligned}
y_{\text{int}}:\ & \text{let } x = 0 \\
& \therefore y = 1 \\
& \therefore (0; 1)
\end{aligned}
$$

$$
\begin{aligned}
y_{\text{int}}:\ & \text{let } y = 0 \\
\frac{1}{8}x^3 + 1 &= 0 \\
x^3 &= -8 \\
x &= -2 \\
\therefore x_{\text{int}}&(-2; 0)
\end{aligned}
$$



3. $h : x \rightarrow (x-2)^3$

   Solution:

   $h(x) = (x-2)^3 = x^3 - 6x^2 + 12x - 8$

   a) Gradient:

$$
\begin{aligned}
h'(x) &= 3x^2 - 12x + 12 \\
&= 3(x^2 - 4x + 4) \\
&= 3(x-2)^2
\end{aligned}
$$

   Gradient always > 0 for all x except $x = 2$ where the gradient will be 0.
   Stationary point: $(2; 0)$

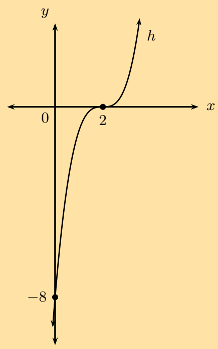
b) Concavity: $h''(x) = 6x - 12$
$h''(x) < 0$ for $x < 2$: concave down
$h''(x) = 0$ for $x = 2$: point of inflection
$h''(x) > 0$ for $x > 2$: concave up

c) Point of inflection: $(2; 0)$

Intercepts:

$y_{\text{int}} : (0; -8)$

$x_{\text{int}} : (2; 0)$



Check answers online with the exercise code below or click on 'show me the answer'.

1. 28Y6 2. 28Y7 3. 28Y8

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Exercise 6 – 10: Mixed exercises on cubic graphs

1. Given $f(x) = x^3 + x^2 - 5x + 3$.

a) Show that $(x - 1)$ is a factor of $f(x)$ and hence factorise $f(x)$.

Solution:

First substitute in $x = 1$ to check if $(x - 1)$ is a factor:

$$\begin{aligned} f(1) &= (1)^3 + (1)^2 - 5(1) + 3 \\ &= 1 + 1 - 5 + 3 \\ &= 0 \end{aligned}$$

Therefore $(x - 1)$ is a factor of $f(x)$.

$$\begin{aligned} f(x) &= (x - 1)(x^2 + 2x - 3) \\ &= (x - 1)(x + 3)(x - 1) \end{aligned}$$

b) Determine the coordinates of the intercepts and the turning points.

Solution:

For the y-intercept, let $x = 0$: $f(0) = 0^3 + 0^2 - 5(0) + 3 = 3$, which gives the point $(0; 3)$.

For the x-intercepts, let $f(x) = 0$:

$$0 = (x - 1)(x + 3)(x - 1)$$

Therefore, the x-intercepts are: $(1; 0)$ and $(-3; 0)$.

Find the turning points:

$$f'(x) = 3x^2 + 2x - 5$$
$$0 = 3x^2 + 2x - 5$$
$$0 = (3x+5)(x-1)$$
$$\therefore x = -\frac{5}{3} \text{ or } x = 1$$

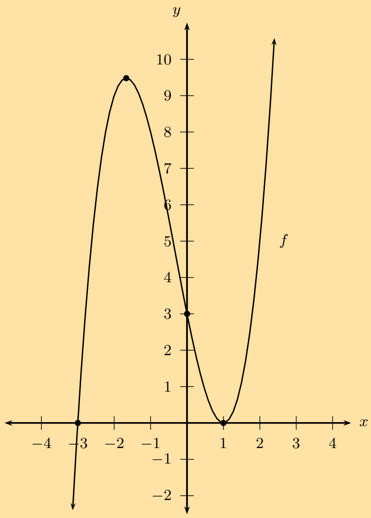
To find the y-values, we substitute these two values for x into the original equation. We already know that when $x = 1$, $y = 0$. Substituting in the other value gives:

$$f\left(-\frac{5}{3}\right) = \left(-\frac{5}{3}\right)^3 + \left(-\frac{5}{3}\right)^2 - 5\left(-\frac{5}{3}\right) + 3$$
$$= \frac{256}{27}$$

Therefore the turning points are: $(1; 0)$ and $\left(-\frac{5}{3}; \frac{256}{27}\right)$

c) Sketch the graph.

Solution:



2. a) Sketch the graph of $f(x) = -x^3 + 4x^2 + 11x - 30$. Show all the turning points and intercepts with the axes.

Solution:

We find the y-intercept by letting $x = 0$:

$$f(x) = -x^3 + 4x^2 + 11x - 30$$
$$f(0) = -(0)^3 + 4(0)^2 + 11(0) - 30$$
$$= -30$$

The y-intercept is: $(0; -30)$

We find the x-intercepts by letting $f(x) = 0$.

We use the factor theorem to check if $(x-1)$ is a factor.

$$f(x) = -x^3 + 4x^2 + 11x - 30$$
$$f(1) = -(1)^3 + 4(1)^2 + 11(1) - 30$$
$$= -16$$

Therefore, $(x-1)$ is not a factor.

We now use the factor theorem to check if $(x+1)$ is a factor.

$$\begin{aligned} f(x) &= -x^3 + 4x^2 + 11x - 30 \\ f(-1) &= -(-1)^3 + 4(-1)^2 + 11(-1) - 30 \\ &= -36 \end{aligned}$$

Therefore, $(x + 1)$ is not a factor.
We now try $(x - 2)$:

$$\begin{aligned} f(x) &= -x^3 + 4x^2 + 11x - 30 \\ f(2) &= -(2)^3 + 4(2)^2 + 11(2) - 30 \\ &= 0 \end{aligned}$$

Therefore, $(x - 2)$ is a factor.

$$\begin{aligned} f(x) &= (x - 2)(-x^2 + 2x + 15) \\ &= -(x - 2)(x^2 - 2x - 15) \\ &= -(x - 2)(x + 3)(x - 5) \end{aligned}$$

The x-intercepts are: $(2; 0), (-3; 0), (5; 0)$.
For the turning points, let $f'(x) = 0$.

$$\begin{aligned} f'(x) &= -3x^2 + 8x + 11 \\ &= -(3x^2 - 8x - 11) \\ \therefore 0 &= (3x - 11)(x + 1) \end{aligned}$$

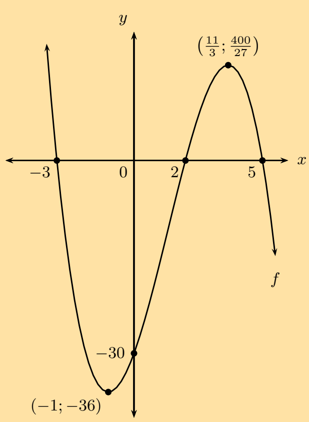
The x-coordinates of the turning points are: $x = -1$ and $x = \frac{11}{3}$.
The y-coordinates of the turning points are calculated as:

$$\begin{aligned} f(-1) &= -(-1)^3 + 4(-1)^2 + 11(-1) - 30 \\ &= -36 \end{aligned}$$

and

$$\begin{aligned} f(\frac{11}{3}) &= -\left(\frac{11}{3}\right)^3 + 4\left(\frac{11}{3}\right)^2 + 11\left(\frac{11}{3}\right) - 30 \\ &= \frac{400}{3} \end{aligned}$$

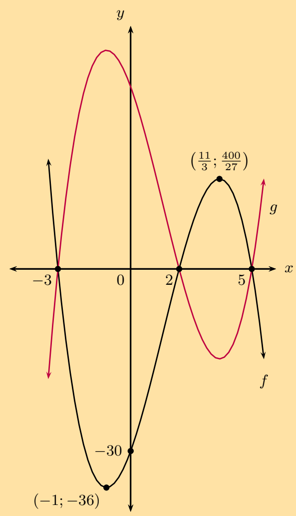
Therefore, the turning points are: $(-1; -36)$ and $(\frac{11}{3}; \frac{400}{27})$.



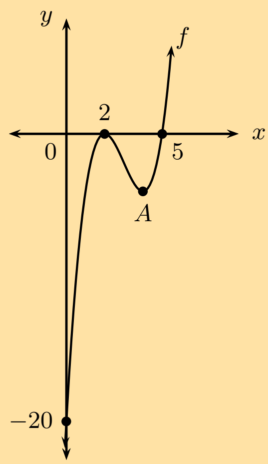
b) Given $g(x) = x^3 - 4x^2 - 11x + 30$, sketch the graph of g without any further calculations. Describe the method for drawing the graph.

Solution:

It is the mirror image of f in the x-axis. In other words, we reflect the graph of f about the x-axis.



3. The sketch shows the graph of a cubic function, f, with a turning point at $(2; 0)$, going through $(5; 0)$ and $(0; -20)$.



a) Find the equation of f.

Solution:

x-intercepts are $(2; 0)$, $(2; 0)$ and $(5; 0)$.

$$\begin{aligned} \therefore \text{ Equation of } f: \quad y &= a(x-2)^2(x-5) \\ &= a(x^2-4x+4)(x-5) \\ &= a(x^3-9x^2+24x-20) \\ &= ax^3-9ax^2+24ax-20a \end{aligned}$$

From the graph, y_{int} is $(0; 20)$.

$$\begin{aligned} \therefore -20a &= -20 \\ a &= 1 \\ \therefore f: \quad y &= x^3-9x^2+24x-20 \end{aligned}$$

b) Find the coordinates of turning point A.

Solution:

The turning points are where $f'(x) = 0$:

$$\begin{aligned} f'(x) &= 3x^2 - 18x + 24 \\ 3x^2 - 18x + 24 &= 0 \\ (3x - 12)(x - 2) &= 0 \\ \therefore 3x - 12 = 0 &\text{ or } x - 2 = 0 \\ \therefore x = 4 &\text{ or } x = 2 \\ \therefore \text{ Point } A &\text{ is } (4; y) \end{aligned}$$

Substitute $x = 4$ into the equation of f to calculate corresponding y-value:

$$\begin{aligned} y &= (4)^3 - (9)(4)^2 + 24(4) - 20 \\ &= 64 - 144 + 96 - 20 \\ &= -4 \\ \therefore \text{ Point } A \text{ is } (4; -4) \end{aligned}$$

4. a) Find the intercepts and stationary point(s) of $f(x) = -\frac{1}{3}x^3 + 2$ and draw a sketch of the graph.

Solution:

$$\begin{aligned} y_{\text{int}}: \text{ Let } x &= 0 \\ \therefore y &= -\frac{1}{3}(0)^3 + 2 \\ y &= 2 \end{aligned}$$

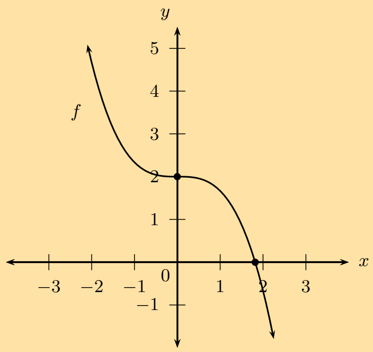
This gives the point $(0; 2)$.

$$\begin{aligned} x_{\text{int}}: \text{ Let } y &= 0 \\ -\frac{1}{3}x^3 + 2 &= 0 \\ -\frac{1}{3}x^3 &= -2 \\ x^3 &= 6 \\ x &= \sqrt[3]{6} \end{aligned}$$

This gives the point $\left(\sqrt[3]{6}; 0\right)$.

$$\begin{aligned} \text{Stationary points: where } f'(x) &= 0 \\ f'(x) &= -x^2 \\ -x^2 &= 0 \\ \therefore x &= 0 \\ \text{Substitute into } f \therefore y &= -\frac{1}{3}(0)^2 + 2 \\ y &= 2 \end{aligned}$$

There is only one stationary point at $(0; 2)$. From $f'(x) = -x^2$, we know that the gradient of the function is always negative, therefore this is a point of inflection.



b) For which values of x will:

i. $f(x) < 0$
ii. $f'(x) < 0$
iii. $f''(x) < 0$

Motivate each answer.

Solution:

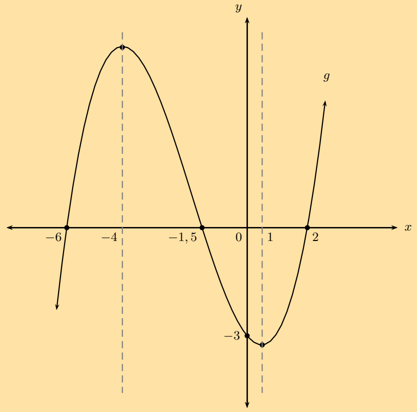
i. For $f(x) < 0$, the function values are negative and this is true for $x > \sqrt[3]{6}$.
ii. For $f'(x) < 0$, the gradient of $f(x)$ is negative, so where $f'(x) < 0$, and this is true for $x \in R$, $x \neq 0$.
iii. $f''(x) < 0$ where $f(x)$ is concave down and this is true where $x > 0$.

5. Use the information below to sketch a graph of each cubic function (do not find the equations of the functions).

a)

$$g(-6) = g(-1{,}5) = g(2) = 0$$
$$g'(-4) = g'(1) = 0$$
$$g'(x) > 0 \text{ for } x < -4 \text{ or } x > 1$$
$$g'(x) < 0 \text{ for } -4 < x < 1$$

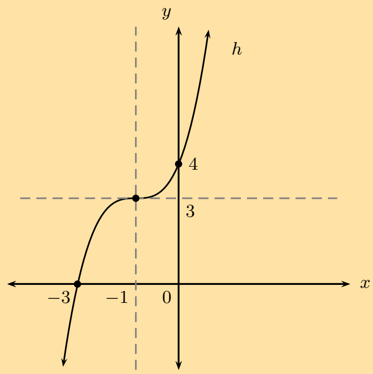
Solution:



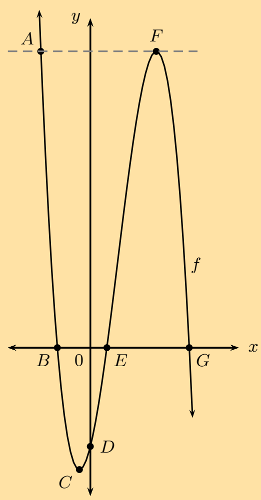
b)

$$h(-3) = 0$$
$$h(0) = 4$$
$$h(-1) = 3$$
$$h'(-1) = 0$$
$$h''(-1) = 0$$
$$h'(x) > 0 \text{ for all } x \text{ values except } x = -1$$

Solution:



6. The sketch below shows the curve of $f(x) = -(x+2)(x-1)(x-6)$ with turning points at C and F. AF is parallel to the x-axis.



Determine the following:

a) length OB

Solution:

$$\text{Find } x_{\text{int}} \text{ of } f\text{: let } y = 0$$
$$\therefore -(x+2)(x-1)(x-6) = 0$$
$$x = -2 \ (B), \quad x = 1 \ (E), \quad x = 6 \ (G)$$
$$\therefore OB = 2 \text{ units} \quad \text{(Note length is always positive)}$$

b) length OE

Solution:

$OE = 1$ unit

c) length EG

Solution:

$EG = 6 - 1 = 5$ units

d) length OD

Solution:

Find y_{int} by writing f in expanded form:

$$\begin{aligned} y &= -(x+2)(x^2 - 7x + 6) \\ &= -(x^3 - 5x^2 - 8x + 12) \\ &= -x^3 + 5x^2 + 8x - 12 \end{aligned}$$
$$\therefore OD = 12 \text{ units}$$

e) coordinates of C and F

Solution:

C and F are the turning points.

To find the x-coordinates of C and F, find $f'(x)$:

$$\begin{aligned} f'(x) &= -3x^2 + 10x + 8 \\ \text{At turning points, } f'(x) &= 0 \\ \therefore -3x^2 + 10x + 8 &= 0 \\ 3x^2 - 10x - 8 &= 0 \\ (3x+2)(x-4) &= 0 \\ x = -\frac{2}{3} &\text{ or } x = 4 \end{aligned}$$

Substitute to find y-coordinates of the turning points:

$$\begin{aligned} x = -\frac{2}{3}: \quad y &= -\left(-\frac{2}{3}\right)^3 + 5\left(-\frac{2}{3}\right)^2 + 8\left(-\frac{2}{3}\right) - 12 \\ &= \frac{8}{27} + \frac{20}{9} - \frac{16}{3} - 12 \\ &= \frac{8 + 60 - 144 - 324}{27} \\ &= -\frac{400}{27} \\ &\approx -14{,}8 \\ \therefore C &= \left(-\frac{2}{3}; -\frac{400}{27}\right) \\ x = 4: \quad y &= -(4)^3 + 5(4)^2 + 8(4) - 12 \\ &= -64 + 80 + 32 - 12 \\ &= 36 \\ \therefore F &= (4; 36) \end{aligned}$$

f) length AF

Solution:

A has same y-coordinate as F, $y = 36$. Therefore $-x^3 + 5x^2 + 8x - 12 = 36$ at F.

Solve for x to find the x-coordinate of A: $x^3 - 5x^2 - 8x + 48 = 0$.

We know that $x = 4$ is a solution of this equation, therefore $(x - 4)$ is a factor.

$$\begin{aligned} \therefore x^3 - 5x^2 - 8x + 48 &= (x-4)(x^2 - x - 12) = 0 \\ \text{Solve } x^2 - x - 12 &= 0 \\ (x-4)(x+3) &= 0 \\ \therefore x = 4, \quad x &= -3 \\ \therefore x\text{-coordinate of } A \text{ is } &-3 \\ x\text{-coordinate of } F \text{ is } &4 \\ \therefore AF &= 7 \text{ units} \end{aligned}$$

g) average gradient between E and F

Solution:

$$\begin{aligned} \text{Average gradient} &= \frac{f(4) - f(1)}{4 - 1} \\ &= \frac{36 - 0}{3} \\ &= 12 \end{aligned}$$

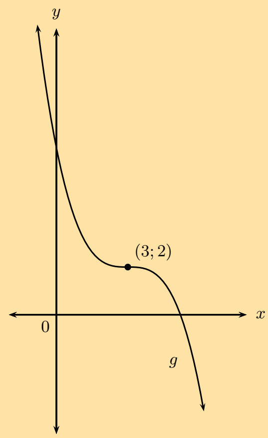
h) the equation of the tangent to the graph at E

Solution:

$$\text{Gradient at } E = f'(1)$$
$$f'(x) = -3x^2 + 10x + 8$$
$$f'(1) = -3(1)^2 + 10(1) + 8$$
$$= 15$$
$$\therefore y = 15x + c$$
$$\text{Tangent goes through } (1;0)$$
$$\therefore 0 = 15(1) + c$$
$$\therefore c = -15$$

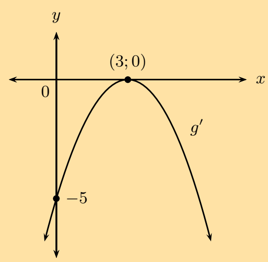
Therefore the equation of the tangent is $y = 15x - 15$.

7. Given the graph of a cubic function with the stationary point $(3;2)$, sketch the graph of the derivative function if it is also given that the gradient of the graph is -5 at $x = 0$.



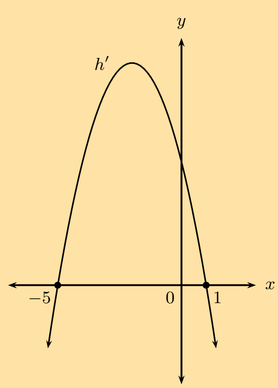
Solution:

Derivative will be of the second degree: parabola. Gradient of function is negative throughout, except at $(3;2)$, where gradient $= 0$, $\therefore g'(3) = 0$. g' has a maximum value of 0 where $x = 3$. Therefore $g'(0) = -5$.



8. The sketch below shows the graph of $h'(x)$ with x-intercepts at -5 and 1.

Draw a sketch graph of $h(x)$ if $h(-5) = 2$ and $h(1) = 6$.

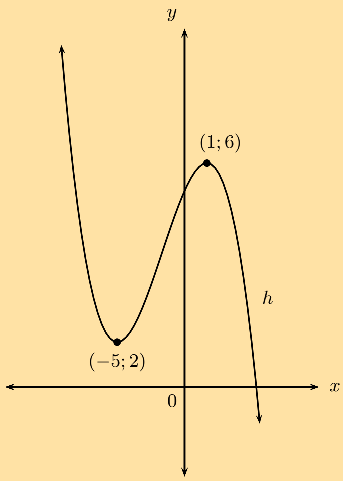


Solution:

$h(x)$ is a cubic function because $h'(x)$ is a parabola. $h(x)$ has two turning points because $h'(x)$ has two x-intercepts.

The x-values of the turning points of $h(x)$ are the x-intercepts of $h'(x)$, where $h'(x) = 0$.

So $h(x)$ will have turning points at $(-5; 2)$: minimum turning point (where gradient changes from negative to positive) and at $(1; 6)$: maximum turning point (where gradient changes from positive to negative).



Check answers online with the exercise code below or click on 'show me the answer'.

1. 28Y9 2. 28YB 3. 28YC 4. 28YD 5a. 28YF 5b. 28YG
6. 28YH 7. 28YJ 8. 28YK

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6.7 Applications of differential calculus

Optimisation problems

Exercise 6 – 11: Solving optimisation problems

1. The sum of two positive numbers is 20. One of the numbers is multiplied by the square of the other. Find the numbers that make this product a maximum.

   Solution:

   Let the first number be x and the second number be y and let the product be P. We get the following two equations:

   $$x + y = 20$$
   $$xy^2 = P$$

   Rearranging the first equation and substituting into the second gives:

   $$P = (20 - x)^2 x$$
   $$= 400x - 40x^2 + x^3$$

Differentiating and setting to 0 gives:

$$\begin{aligned} P' &= 400 - 80x + 3x^2 \\ 0 &= 3x^2 - 80x + 400 \\ &= (3x - 20)(x - 20) \end{aligned}$$

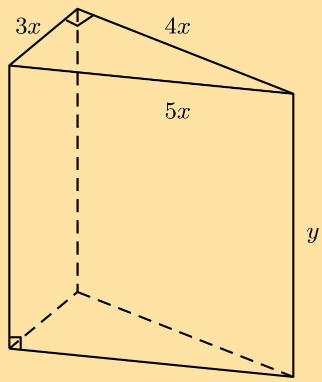
Therefore, $x = 20$ or $x = \frac{20}{3}$.

If $x = 20$ then $y = 0$ and the product is a minimum, not a maximum.

Therefore, $x = \frac{20}{3}$ and $y = 20 - \frac{20}{3} = \frac{40}{3}$.

Therefore the two numbers are $\frac{20}{3}$ and $\frac{40}{3}$ (approximating to the nearest integer gives 7 and 13).

2. A wooden block is made as shown in the diagram. The ends are right-angled triangles having sides $3x$, $4x$ and $5x$. The length of the block is y. The total surface area of the block is 3600 cm^2.



a) Show that $y = \frac{300-x^2}{x}$.

Solution:

We start by finding the surface area of the prism:

$$\begin{aligned} \text{Surface area} &= 2\left(\frac{1}{2}b \times h\right) + 3xy + 4xy + 5xy \\ 3600 &= (3x \times 4x) + 12xy \\ &= 12x^2 + 12xy \end{aligned}$$

Solving for y gives:

$$\begin{aligned} 12xy &= 3600 - 12x^2 \\ y &= \frac{3600 - 12x^2}{12x} \\ y &= \frac{300 - x^2}{x} \end{aligned}$$

b) Find the value of x for which the block will have a maximum volume. (Volume = area of base $\times$ height)

Solution:

Start by finding an expression for volume in terms of x:

$$\begin{aligned} V &= \text{ area of triangle } \times y \\ V &= 6x^2 \times \frac{300 - x^2}{x} \\ &= 6x(300 - x^2) \\ &= 1800x - 6x^3 \end{aligned}$$

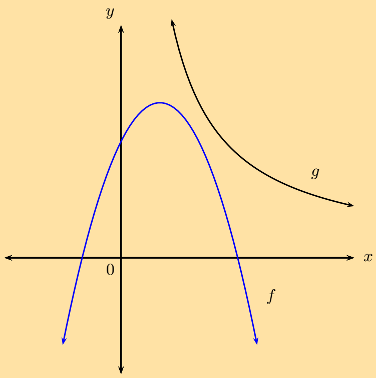
Now take the derivative and set it equal to 0:

$$\begin{aligned} V' &= 1800 - 18x^2 \\ 0 &= 1800 - 18x^2 \\ 18x^2 &= 1800 \\ x^2 &= 100 \\ x &= \pm 10 \end{aligned}$$

Since the length can only be positive, $x = 10$
$\therefore x = 10$ cm

3. Determine the shortest vertical distance between the curves of f and g if it is given that:

$$\begin{aligned} f(x) &= -x^2 + 2x + 3 \\ \text{and } g(x) &= \frac{8}{x}, \quad x > 0 \end{aligned}$$



Solution:

$$\begin{aligned} \text{Let the distance } P(x) &= g(x) - f(x) \\ &= \frac{8}{x} - (-x^2 + 2x + 3) \\ &= \frac{8}{x} + x^2 - 2x - 3 \end{aligned}$$

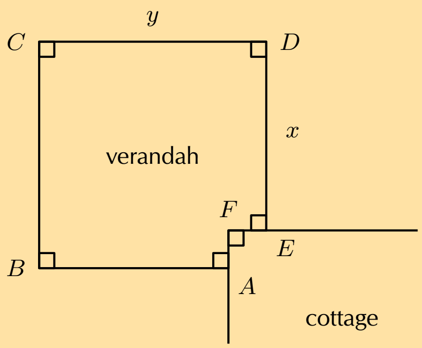
To minimise the distance between the curves, let $P'(x) = 0$:

$$\begin{aligned} P'(x) &= -\frac{8}{x^2} + 2x - 2 \qquad (x \neq 0) \\ 0 &= -\frac{8}{x^2} + 2x - 2 \\ \therefore 0 &= -8 + 2x^3 - 2x \\ 0 &= 2x^3 - 2x - 8 \\ 0 &= x^3 - x - 4 \\ 0 &= (x-2)(x^2 + x + 2) \\ \therefore x = 2 \text{ or } x &= \frac{-1 \pm \sqrt{(1)^2 - 4(1)(2)}}{2(1)} \\ &= \text{ no real solutions} \\ \therefore x &= 2 \end{aligned}$$

Therefore, the shortest distance:

$$P(2) = \frac{8}{(2)} + (2)^2 - 2(2) - 3$$
$$= 4 + 4 - 4 - 3$$
$$= 1 \text{ unit}$$

4. The diagram shows the plan for a verandah which is to be built on the corner of a cottage. A railing $ABCDE$ is to be constructed around the four edges of the verandah.



If $AB = DE = x$ and $BC = CD = y$, and the length of the railing must be 30 m, find the values of x and y for which the verandah will have a maximum area.

Solution:

We need to determine an expression for the area in terms of only one variable.

The perimeter is:

$$P = 2x + 2y$$
$$30 = 2x + 2y$$
$$15 = x + y$$
$$y = 15 - x$$

The area is:

$$A = y^2 - (y - x)^2$$
$$= y^2 - (y^2 - 2xy + x^2)$$
$$= y^2 - y^2 + 2xy - x^2$$
$$= 2xy - x^2$$

We use the expression for perimeter to eliminate the y variable so that we have an expression for area in terms of x only:

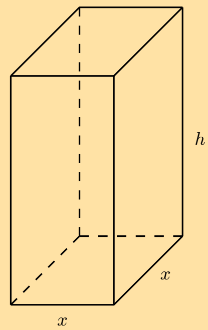
$$A(x) = 2x(15 - x) - x^2$$
$$= 30x - 2x^2 - x^2$$
$$= 30x - 3x^2$$

To find the maximum, we need to take the derivative and set it equal to 0:

$$A'(x) = 30 - 6x$$
$$0 = 30 - 6x$$
$$6x = 30$$
$$x = 5$$

Therefore, $x = 5$ m and substituting this value back into the formula for perimeter gives $y = 10$ m.

5. A rectangular juice container, made from cardboard, has a square base and holds 750 cm^3 of juice. The container has a specially designed top that folds to close the container. The cardboard needed to fold the top of the container is twice the cardboard needed for the base, which only needs a single layer of cardboard.



a) If the length of the sides of the base is x cm, show that the total area of the cardboard needed for one container is given by:

$$A(\text{in square centimetres}) = \frac{3000}{x} + 3x^2$$

Solution:

$$\begin{aligned} V &= x^2 h \\ 750 &= x^2 h \\ \therefore h &= \frac{750}{x^2} \\ A &= \text{ area of sides } + \text{ area of base } + \text{ area of top} \\ &= 4xh + x^2 + 2x^2 \\ &= 4xh + 3x^2 \\ \text{Substitute } h &= \frac{750}{x^2} : \\ A &= 4x\left(\frac{750}{x^2}\right) + 3x^2 \\ &= \frac{3000}{x} + 3x^2 \end{aligned}$$

b) Determine the dimensions of the container so that the area of the cardboard used is minimised.

Solution:

$$\begin{aligned} A(x) &= \frac{3000}{x} + 3x^2 \\ A'(x) &= -\frac{3000}{x^2} + 6x \\ \therefore 0 &= -\frac{3000}{x^2} + 6x \\ 6x &= \frac{3000}{x^2} \\ x^3 &= 500 \\ \therefore x &= \sqrt[3]{500} \\ &\approx 7{,}9 \text{ cm} \\ \therefore h &= \frac{750}{(7{,}9)^2} \\ &\approx 12{,}0 \text{ cm} \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.
1. 28YN 2a. 28YP 2b. 28YQ 3. 28YR 4. 28YS 5a. 28YT
5b. 28YV

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Rates of change

Exercise 6 – 12: Rates of change

1. A pump is connected to a water reservoir. The volume of the water is controlled by the pump and is given by the formula:

$$V(d) = 64 + 44d - 3d^2$$
where $V =$ volume in kilolitres
$d =$ days

a) Determine the rate of change of the volume of the reservoir with respect to time after 8 days.
Solution:

$$\text{Rate of change} = V'(d)$$
$$V'(d) = 44 - 6d$$
After 8 days, rate of change will be:
$$V'(8) = 44 - 6(8)$$
$$= -4 \text{ k}\ell \text{ per day}$$

b) Is the volume of the water increasing or decreasing at the end of 8 days. Explain your answer.
Solution:
The rate of change is negative, so the function is decreasing.

c) After how many days will the reservoir be empty?
Solution:

$$\text{Reservoir empty: } V(d) = 0$$
$$\therefore 64 + 44d - 3d^2 = 0$$
$$(16 - d)(4 + 3d) = 0$$
$$\therefore d = 16 \text{ or } d = -\frac{4}{3}$$
$$\therefore \text{ It will be empty after 16 days}$$

d) When will the amount of water be at a maximum?
Solution:
Maximum at turning point.

$$\text{Turning point where } V'(d) = 0$$
$$\therefore 44 - 6d = 0$$
$$d = \frac{44}{6}$$
$$= 7\frac{1}{3} \text{ days}$$

e) Calculate the maximum volume.

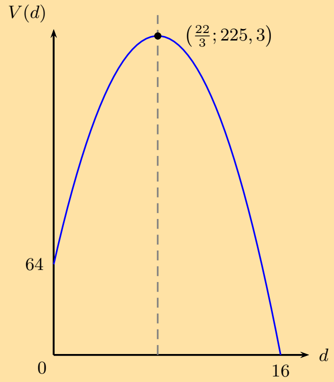
Solution:

Maximum at turning point.

$$\text{Maximum volume } = V\left(\frac{22}{3}\right)$$
$$V\left(\frac{22}{3}\right) = 64 + 44\left(\frac{22}{3}\right) - 3\left(\frac{22}{3}\right)^2$$
$$= 225{,}3 \text{ k}\ell$$

f) Draw a graph of $V(d)$.

Solution:



2. A soccer ball is kicked vertically into the air and its motion is represented by the equation:

$$D(t) = 1 + 18t - 3t^2$$
where $D =$ distance above the ground (in metres)
$t =$ time elapsed (in seconds)

a) Determine the initial height of the ball at the moment it is being kicked.

Solution:

$$D(t) = 1 + 18t - 3t^2$$
$$D(0) = 1 + 18(0) - 3(0)^2$$
$$= 1 \text{ metre}$$

b) Find the initial velocity of the ball.

Solution:

$$\text{Velocity } = D'(t) = 18 - 6t$$
$$\text{Initial velocity } = D'(0)$$
$$D'(0) = 18 - 6(0) = 18 \text{ m.s}^{-1}$$

c) Determine the velocity of the ball after 1,5 s.

Solution:

$$\text{Velocity after } 1{,}5 \text{ s} = D'(1{,}5)$$
$$D'(1{,}5) = 18 - 6(1{,}5)^2$$
$$= 18 - 9$$
$$= 9 \text{ m.s}^{-1}$$

d) Calculate the maximum height of the ball.

Solution:

Maximum height is at the turning point.

$$\begin{aligned}
\text{Turning point: } D'(t) &= 0\\
\therefore 18 - 6t &= 0\\
6t &= 18\\
t &= 3 \text{ s}\\
\text{Maximum height } &= D(3)\\
&= 1 + (18)(3) - (3)(3)^2\\
&= 28 \text{ metres}
\end{aligned}$$

e) Determine the acceleration of the ball after 1 second and explain the meaning of the answer.

Solution:

$$\begin{aligned}
\text{Acceleration } &= D''(t)\\
D''(t) &= -6 \text{ m.s}^{-2}
\end{aligned}$$

Interpretation: the velocity is decreasing by 6 metres per second per second.

f) Calculate the average velocity of the ball during the third second.

Solution:

Average velocity during third second:

$$\begin{aligned}
&= \frac{D(3) - D(2)}{3 - 2}\\
&= \frac{1 + 18(3) - 3(3)^2 - \left[1 + 18(2) - 3(2)^2\right]}{1}\\
&= 3 \text{ m.s}^{-1}
\end{aligned}$$

g) Determine the velocity of the ball after 3 seconds and interpret the answer.

Solution:

$$\begin{aligned}
\text{Instantaneous velocity} &= D'(3)\\
&= 18 - 6(3)\\
&= 0 \text{ m.s}^{-1}
\end{aligned}$$

Interpretation: this is the stationary point, where the derivative is zero. The ball has stopped going up and is about to begin its descent.

h) How long will it take for the ball to hit the ground?

Solution:

$$\begin{aligned}
\text{Hits ground: } D(t) &= 0\\
-3t^2 + 18t + 1 &= 0\\
t &= \frac{-18 \pm \sqrt{(18^2 - 4(1)(-3)}}{2(-3)}\\
t &= \frac{-18 \pm \sqrt{336}}{-6}\\
\therefore t &= -0{,}05 \text{ or } t = 6{,}05
\end{aligned}$$

The ball hits the ground at $6{,}05$ s (time cannot be negative).

i) Determine the velocity of the ball when it hits the ground.

Solution:

$$\text{Velocity after } 6{,}05 \text{ s} = D'(6{,}05)$$
$$D'(6{,}05) = 18 - 5(6{,}05) = -18{,}3 \text{ m.s}^{-1}$$

3. If the displacement s (in metres) of a particle at time t (in seconds) is governed by the equation $s = \frac{1}{2}t^3 - 2t$, find its acceleration after 2 seconds.

   Solution:
   We know that velocity is the rate of change of displacement. This means that $\frac{dS}{dt} = v$:

   $$s = \frac{1}{2}t^3 - 2t$$
   $$v = \frac{3}{2}t^2 - 2$$

   We also know that acceleration is the rate of change of velocity. This means that $\frac{dv}{dt} = a$:

   $$v = \frac{3}{2}t^2 - 2$$
   $$a = 3t$$

   Substituting $t = 2$ gives $a = 6$ m.s^{-2}.

4. During an experiment the temperature T (in degrees Celsius) varies with time t (in hours) according to the formula: $T(t) = 30 + 4t - \frac{1}{2}t^2, \ \ t \in [1; 10]$.

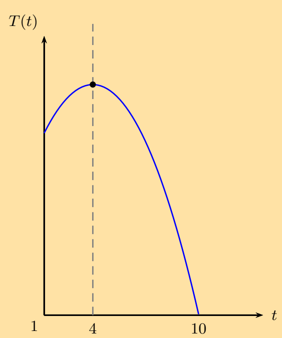
   a) Determine an expression for the rate of change of temperature with time.
   Solution:
   We find the rate of change of temperature with time by differentiating:

   $$T(t) = 30 + 4t - \frac{1}{2}t^2$$
   $$T'(t) = 4 - t$$

   b) During which time interval was the temperature dropping?
   Solution:
   We set the derivative equal to 0:

   $$0 = 4 - t$$
   $$t = 4$$

   We look at the coefficient of the t^2 term to decide whether this is a minimum or maximum point. The coefficient is negative and therefore the function must have a maximum value. The interval in which the temperature is increasing is $[1; 4)$. The interval in which the temperature is dropping is $(4; 10]$. We can check this by drawing the graph or by substituting in the values for t into the original equation.



Check answers online with the exercise code below or click on 'show me the answer'.

1. 28YW 2a. 28YX 2b. 28YY 2c. 28YZ 2d. 28Z2 2e. 28Z3
2f. 28Z4 2g. 28Z5 2h. 28Z6 2i. 28Z7 3. 28Z8 4a. 28Z9
4b. 28ZB

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Exercise 6 – 13: End of chapter exercises

1. Determine $f'(x)$ from first principles if $f(x) = 2x - x^2$.

   Solution:

$$\begin{aligned} f(x) &= 2x - x^2 \\ f(x+h) &= -(x+h)^2 + 2(x+h) \\ f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{-(x+h)^2 + 2(x+h) - (-x^2 + 2x)}{h} \\ &= \lim_{h\to 0} \frac{-x^2 - 2xh - h^2 + 2x + 2h + x^2 - 2x}{h} \\ &= \lim_{h\to 0} \frac{-2xh - h^2 + 2h}{h} \\ &= \lim_{h\to 0} \frac{h(-2x - h + 2)}{h} \\ &= \lim_{h\to 0} (-2x - h + 2) \\ &= -2x + 2 \end{aligned}$$

2. Given $f(x) = \frac{1}{x} + 3$, find $f'(x)$ using the definition of the derivative.

   Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\left(\frac{1}{x+h} + 3\right) - \left(\frac{1}{x} + 3\right)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1+3(x+h)}{x+h} - \frac{1+3x}{x}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x(1+3x+3h) - (x+h)(1+3x)}{x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{x(1 + 3x + 3h) - (x+h)(1+3x)}{h\ .\ x(x+h)} \\ &= \lim_{h\to 0} \frac{x + 3x^2 + 3xh - x - 3x^2 - h - 3xh}{hx(x+h)} \\ &= \lim_{h\to 0} \frac{-h}{hx(x+h)} \\ &= \lim_{h\to 0} \frac{-1}{x^2 + xh} \\ &= -\frac{1}{x^2} \end{aligned}$$

3. Calculate: $\lim_{x\to 1} \frac{1 - x^3}{1 - x}$

   Solution:

$$\begin{aligned} \lim_{x\to 1} \frac{x^3 - 1}{x - 1} &= \lim_{x\to 1} \frac{(x-1)(x^2 + x + 1)}{x - 1} \\ &= \lim_{x\to 1} (x^2 + x + 1) \\ &= 3 \end{aligned}$$

4. Determine $\frac{dy}{dx}$ if:

a) $y = (x+2)(7-5x)$

Solution:

$$
\begin{aligned}
y &= 14 - 3x - 5x^2 \\
\frac{dy}{dx} &= -3 - 10x
\end{aligned}
$$

b) $y = \frac{8x^3+1}{2x+1}$

Solution:

$$
\begin{aligned}
y &= \frac{8x^3+1}{2x+1} \\
&= \frac{(2x+1)(4x^2-2x+1)}{2x+1} \\
&= 4x^2 - 2x + 1 \\
\frac{dy}{dx} &= 8x - 2
\end{aligned}
$$

c) $y = (2x)^2 - \frac{1}{3x}$

Solution:

$$
\begin{aligned}
f(x) &= (2x)^2 - \frac{1}{3x} \\
f(x) &= 4x^2 - \frac{x^{-1}}{3} \\
f'(x) &= 8x + \frac{x^{-2}}{3} \\
f'(x) &= 8x + \frac{1}{3x^2}
\end{aligned}
$$

d) $y = \frac{2\sqrt{x}-5}{\sqrt{x}}$

Solution:

$$
\begin{aligned}
f(x) &= \frac{2\sqrt{x}-5}{\sqrt{x}} \\
&= \frac{2x^{\frac{1}{2}}}{x^{\frac{1}{2}}} - \frac{5}{x^{\frac{1}{2}}} \\
&= 2 - 5x^{\frac{-1}{2}} \\
f'(x) &= -5(\frac{-1}{2})x^{\frac{-3}{2}} \\
&= \frac{5}{2\sqrt{x^3}}
\end{aligned}
$$

5. Given: $f(x) = 2x^2 - x$

a) Use the definition of the derivative to calculate $f'(x)$.

Solution:

$$\begin{aligned} f(x) &= 2x^2 - x \\ f(x+h) &= 2(x+h)^2 - (x+h) \\ &= 2(x^2 + 2xh + h^2) - x - h \\ &= 2x^2 + 4xh + 2h^2 - x - h \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(2x^2 + 4xh + 2h^2 - x - h) - (2x^2 - x)}{h} \\ &= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - x - h - 2x^2 + x}{h} \\ &= \lim_{h \to 0} \frac{4xh + 2h^2 - h}{h} \\ &= \lim_{h \to 0} \frac{h(4x + 2h - 1)}{h} \\ &= \lim_{h \to 0} (4x + 2h - 1) \\ &= 4x - 1 \end{aligned}$$

b) Hence, calculate the coordinates of the point at which the gradient of the tangent to the graph of f is 7.

Solution:

$$\begin{aligned} m &= 4x - 1 = 7 \\ 0 &= 4x - 8 \\ 0 &= x - 2 \\ \therefore x &= 2 \end{aligned}$$

The y value is:

$$\begin{aligned} f(x) &= 2x^2 - x \\ \therefore f(2) &= 2(2)^2 - 2 \\ &= 8 - 2 \\ &= 6 \end{aligned}$$

The coordinates of the point are $(2; 6)$.

6. If $g(x) = \left(x^{-2} + x^2\right)^2$, calculate $g'(2)$.

Solution:

$$\begin{aligned} g(x) &= \left(x^{-2} + x^2\right)\left(x^{-2} + x^2\right) \\ &= x^{-4} + x^0 + x^0 + x^4 \\ &= x^{-4} + 1 + 1 + x^4 \\ &= x^{-4} + 2 + x^4 \\ g'(x) &= -4x^{-5} + 4x^3 \\ g'(2) &= -4(2)^{-5} + 4(2)^3 \\ &= \frac{-4}{32} + 32 \\ &= -\frac{1}{8} + 32 \\ &= 31\frac{7}{8} \\ &= \frac{255}{8} \end{aligned}$$

7. Given: $f(x) = 2x - 3$

   a) Find $f^{-1}(x)$.

      Solution:

$$\begin{aligned} y &= 2x - 3 \\ f^{-1}: \quad x &= 2y - 3 \\ 2y &= x + 3 \\ y &= \frac{x}{2} + \frac{3}{2} \end{aligned}$$

   b) Solve $f^{-1}(x) = 3f'(x)$.

      Solution:

$$\begin{aligned} \frac{x+3}{2} &= 3(2) \\ x + 3 &= 12 \\ &= 9 \end{aligned}$$

8. Find the derivative for each of the following:

   a) $p(t) = \frac{\sqrt[5]{t^3}}{3} + 10$

      Solution:

$$\begin{aligned} p(t) &= \frac{\sqrt[5]{t^3}}{3} + 10 \\ p(t) &= \frac{t^{\frac{3}{5}}}{3} + 10 \\ p'(t) &= \frac{1}{3}\left(\frac{3}{5}t^{-\frac{2}{5}}\right) \\ &= \frac{1}{5\sqrt[5]{t^2}} \end{aligned}$$

   b) $k(n) = \frac{(2n^2-5)(3n+2)}{n^2}$

      Solution:

$$\begin{aligned} k(n) &= \frac{6n^3 + 4n^2 - 15n - 10}{n^2} \\ &= 6n + 4 - 15n^{-1} - 10n^{-2} \\ k'(n) &= 6 + 15n^{-2} + 20n^{-3} \\ &= 6 + \frac{15}{n^2} + \frac{20}{n^3} \end{aligned}$$

9. If $xy - 5 = \sqrt{x^3}$, determine $\frac{dy}{dx}$.

   Solution:

$$\begin{aligned} xy - 5 &= \sqrt{x^3} \\ y &= \frac{x^{\frac{3}{2}} + 5}{x} \\ &= x^{\frac{1}{2}} + 5x^{-1} \\ \frac{dy}{dx} &= \frac{1}{2}x^{-\frac{1}{2}} - 5x^{-2} \\ &= \frac{1}{2\sqrt{x}} - \frac{5}{x^2} \end{aligned}$$

10. Given: $y = x^3$

a) Determine $\frac{dy}{dx}$.
Solution:
Differentiate y with respect to x.

$$\frac{dy}{dx} = 3x^2$$

b) Find $\frac{dx}{dy}$.
Solution:
To differentiate x with respect to y, express x in terms of y:

$$\begin{aligned} x &= \sqrt[3]{y} \\ &= y^{\frac{1}{3}} \\ \frac{dx}{dy} &= \frac{1}{3}y^{-\frac{2}{3}} \\ &= \frac{1}{3\sqrt[3]{y^2}} \end{aligned}$$

c) Show that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$.
Solution:

$$\begin{aligned} \frac{dy}{dx} \times \frac{dx}{dy} &= 3x^2 \times \frac{y^{-\frac{2}{3}}}{3} \\ \text{But } y &= x^3 \\ \therefore \frac{dy}{dx} \times \frac{dx}{dy} &= 3x^2 \times \frac{(x^3)^{-\frac{2}{3}}}{3} \\ &= x^2 \times x^{-2} \\ &= x^0 \\ &= 1 \end{aligned}$$

11. Given: $f(x) = x^3 - 3x^2 + 4$

a) Calculate $f(-1)$.
Solution:

$$\begin{aligned} f(-1) &= (-1)^3 - 3(-1)^2 + 4 \\ &= 0 \end{aligned}$$

b) Hence, solve $f(x) = 0$.
Solution:
We know that $(x+1)$ is a factor of $f(x)$ because $f(-1) = 0$. We factorise further using inspection:

$$\begin{aligned} f(x) &= (x+1)(x^2 - 4x + 4) \\ &= (x+1)(x-2)(x-2) \\ \therefore \quad f(x) &= (x+1)(x-2)(x-2) \\ 0 &= (x+1)(x-2)(x-2) \\ \therefore \quad x &= -1 \text{ or } x = 2 \end{aligned}$$

c) Determine $f'(x)$.
Solution:

$$\begin{aligned} f'(x) &= 3x^2 - 6x \\ &= 3x(x-2) \end{aligned}$$

d) Sketch the graph of f, showing the coordinates of the turning points and the intercepts on both axes.

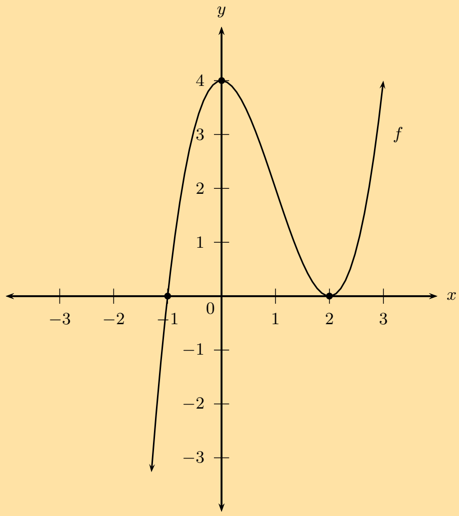
Solution:

The y-intercept is $y = 4$.

The x-intercepts are $(2; 0)$ and $(-1; 0)$.

To find the turning points we let the derivative equal 0. $f'(x) = 3x(x-2) = 0$. The x-values of the turning points are: $x = 0$ and $x = 2$.

Therefore, the turning points are $(0; 4)$ and $(2; 0)$.



e) Determine the coordinates of the points on the graph of f where the gradient is 9.

Solution:

$$
\begin{aligned}
f'(x) &= 3x^2 - 6x \\
3x^2 - 6x &= 9 \\
3x^2 - 6x - 9 &= 0 \\
x^2 - 2x - 3 &= 0 \\
(x-3)(x+1) &= 0 \\
\therefore \quad x &= 3 \text{ or } x = -1 \\
\text{If } x = 3: \quad f(3) &= (3)^3 - 3(3)^2 + 4 \\
&= 27 - 27 + 4 \\
&= 4 \\
\text{If } x = -1: \quad f(-1) &= (-1)^3 - 3(-1)^2 + 4 \\
&= -1 - 3 + 4 \\
&= 0
\end{aligned}
$$

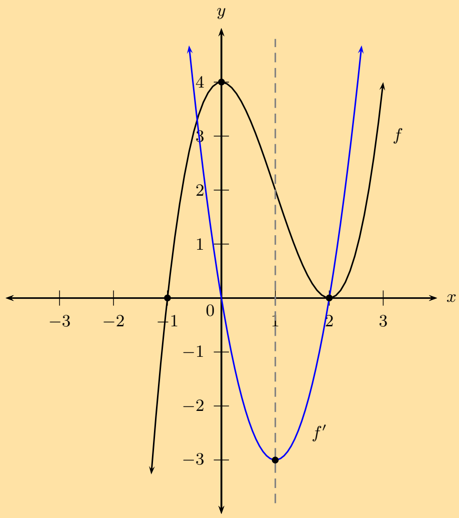
Therefore, at $(-1; 0)$ and $(3; 4)$.

f) Draw the graph of $f'(x)$ on the same system of axes.

Solution:

$$
\begin{aligned}
f'(x) &= 3x^2 - 6x \\
y_{\text{int}}: &\ (0; 0) \\
x_{\text{int}}: &\ 3x(x-2) = 0 \\
\therefore \quad &x = 0 \text{ or } x = 2 \\
\therefore &\ (0; 0) \text{ or } (2; 0)
\end{aligned}
$$

$$
\begin{aligned}
\text{Turning point is where } f''(x) &= 0 \\
\therefore \ 6x - 6 &= 0 \\
\therefore \quad x &= 1 \\
\text{Turning point: } &(1; -3)
\end{aligned}
$$



g) Determine $f''(x)$ and use this to make conclusions about the concavity of f.

Solution:

$$
\begin{aligned}
f''(x) &= 6x - 6 \\
6x - 6 &= 0 \text{ for } x = 1 \\
x < 1, \quad f''(x) &< 0 \therefore f(x) \text{ is concave down} \\
x > 1, \quad f''(x) &> 0 \therefore f(x) \text{ is concave up} \\
x = 1, \quad f''(x) &= 0 \therefore \ f(x) \text{ is point of inflection}
\end{aligned}
$$

12. Given $f(x) = 2x^3 - 5x^2 - 4x + 3$.

a) If $f(-1) = 0$, determine the x-intercepts of f.

Solution:

$$
\begin{aligned}
f(x) &= 2x^3 - 5x^2 - 4x + 3 \\
f(-1) &= 0 \text{, so } (x+1) \text{ is a factor of } f(x) \\
f(x) &= (x+1)(2x^2 - 7x + 3) \\
&= (x+1)(2x-1)(x-3) \\
\text{Let} \quad f(x) &= 0 \\
\therefore \ (x+1)(2x-1)(x-3) &= 0 \\
\therefore \quad x = -1, \quad x &= \frac{1}{2} \text{ or } x = 3
\end{aligned}
$$

The x-intercepts of f are $(-1; 0)$, $\left(\frac{1}{2}; 0\right)$ and $(3; 0)$.

b) Determine the coordinates of the turning points of f.

Solution:

$$
\begin{aligned}
f'(x) &= 6x^2 - 10x - 4 \\
0 &= 2(3x^2 - 5x - 2) \\
&= 3x^2 - 5x - 2 = 0 \\
&= (3x+1)(x-2) = 0 \\
\therefore \quad x &= -\frac{1}{3} \text{ or } x = 2
\end{aligned}
$$

Substitute $x = -\frac{1}{3}$:

$$
\begin{aligned}
y &= 2\left(-\frac{1}{3}\right)^3 - 5\left(-\frac{1}{3}\right)^2 - 4\left(-\frac{1}{3}\right) + 3 \\
&= \frac{100}{27} \\
&= 3{,}7
\end{aligned}
$$

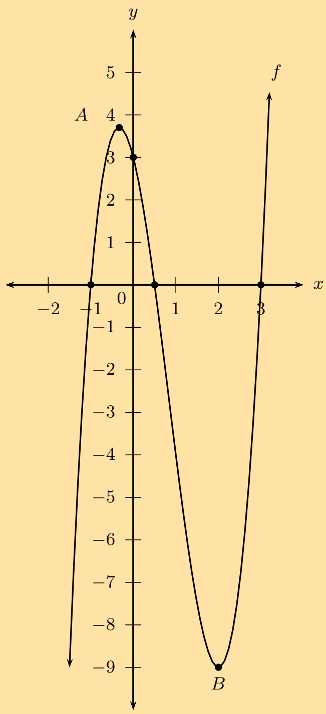
Substitute $x = 2$:

$$
\begin{aligned}
y &= 2(2)^3 - 5(2)^2 - 4(2) + 3 \\
&= -9
\end{aligned}
$$

$\therefore$ Turning points are $A\left(-\frac{1}{3}; 3{,}7\right)$ and $B\,(2; -9)$

c) Draw a sketch graph of f. Clearly indicate the coordinates of the turning points and the intercepts with the axes.

Solution:



d) For which value(s) of k will the equation $f(x) = k$ have three real roots of which two are equal?

Solution:

To find the point when the cubic function has two real roots, we need to find the points where one of the turning points lies on the x-axis. We look at the y-values of the turning points to determine this. For the maximal turning point we lower the graph and for the minimal one we raise the graph. This gives: $k = 3{,}7$ or $k = -9$

e) Determine the equation of the tangent to the graph of $f(x) = 2x^3 - 5x^2 - 4x + 3$ at the point where $x = 1$.

Solution:

Find the y-value at $x = 1$:

$$f(1) = 2(1)^3 - 5(1)^2 - 4(1) + 3$$
$$= -4$$

Determine the value of m substituting $x = 1$ into the derivative:

$$f'(x) = 6x^2 - 10x - 4$$
$$f'(1) = 6(1)^2 - 10(1) - 4$$
$$m = -8$$

The equation of the tangent is:

$$y - y_1 = m(x - x_1)$$
$$y + 4 = -8(x - 1)$$
$$y = -8x + 4$$

13. Given the function $f(x) = x^3 + bx^2 + cx + d$ with y-intercept $(0; 26)$, x-intercept $(-2; 0)$ and a point of inflection at $x = -3$.

a) Show by calculation that $b = 9$, $c = 27$ and $d = 26$.
Solution:

$$y_{\text{int}}(0; 26): \ d = 26$$
$$\therefore f(x) = x^3 + bx^2 + cx + 26$$
$$x_{\text{int}}(-2; 0): \ f(-2) = (-2)^3 + b(-2)^2 + c(-2) + 26$$
$$\therefore 0 = -8 + 4b - 2c + 26$$
$$0 = 4b - 2c + 18$$
$$2b - c = -9 \quad ①$$

Stationary point at $x = -3$

$$\therefore f'(-3) = 0$$
$$f'(x) = 3x^2 + 2bx + c$$
$$f'(-3) = 3(-3)^2 + 2b(-3) + c$$
$$0 = 27 - 6b + c$$
$$6b - c = 27 \quad ②$$
$$2b - c = -9 \quad ①$$
$$① - ②: \ 4b = 36$$
$$b = 9$$

Substitute $b = 9$ into ①

$$2(9) - c = -9$$
$$-c - 27$$
$$c = 27$$
$$\therefore f(x) = x^3 + 9x^2 + 27x + 26$$

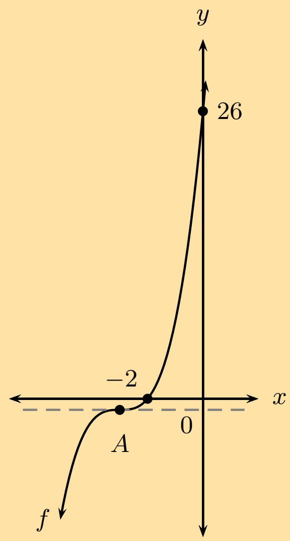
b) Find the y-coordinate of the point of inflection.
Solution:

Passes through $(-3; y)$

$$\therefore y = (-3)^3 + 9(-3)^2 + 27(-3) + 26$$
$$= -27 + 81 - 81 + 26$$
$$= -1$$

$\therefore$ Point of inflection: $A(-3; -1)$

c) Draw the graph of f.
Solution:



d) Discuss the gradient of f.

Solution:

Gradient always positive except at $A(-3;-1)$ where the gradient is at its minimum value, zero.

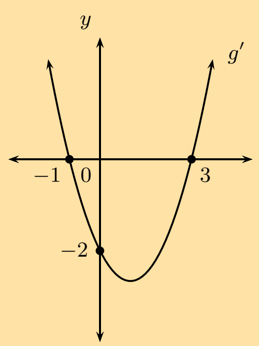
e) Discuss the concavity of f.

Solution:

Concavity changes at inflection point $A(-3;-1)$:

$$f'(x) = 3x^2 + 18x + 27$$
$$f''(x) := 6x + 18$$
$$x < -3, \quad f''(x) < 0 \therefore f(x) \text{ is concave down}$$
$$x = -3, \quad f''(x) = 0 \therefore \quad f(x) \text{ is point of inflection}$$
$$x > 1, \quad f''(x) > 0 \therefore f(x) \text{ is concave up}$$

14. The sketch shows the graph of $g'(x)$.



a) Identify the stationary points of the cubic function, $g(x)$.

Solution:

Turning points are at $x = -1$ and $x = 3$:

For $x = -1$: this is a local maximum turning point (the gradient of g changes from positive to negative).

For $x = 3$: this is a local minimum turning point (the gradient of g changes from negative to positive).

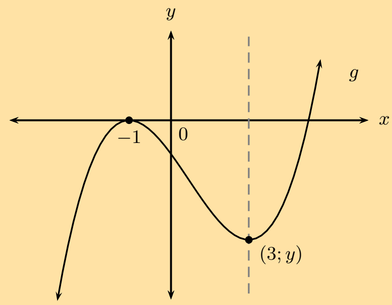
b) What is the gradient of function g where $x = 0$.

Solution:

At $x = 0$, the gradient of the function is equal to -2. We can write this as $g'(0) = -2$.

c) If it is further given that $g(x)$ has only two real roots, draw a rough sketch of $g(x)$. Intercept values do not need to be shown.

Solution:



15. Given that $h(x)$ is a linear function with $h(2) = 11$ and $h'(2) = -1$, find the equation of $h(x)$.

    Solution:

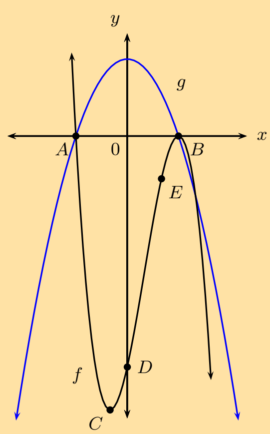
    $h(x)$ is a linear function and is therefore of the form $y = mx + c$. The derivative of a linear function is $\frac{dy}{dx} = m$, therefore from $h'(2) = -1$, we know that $m = -1$.

    $$\begin{aligned} &\therefore y = -x + c \\ \text{Substitute } (2; 11): \quad & 11 = -2 + c \\ &\therefore c = 13 \\ &\therefore y = -x + 13 \end{aligned}$$

    Therefore, $h(x) = -x + 13$.

16. The graphs of f and g and the following points are given below:

    $$A(-3; 0) \quad B(3; 0) \quad C(-1; -32) \quad D(0; -27) \quad E(2; y)$$



a) Use the graphs and determine the values of x for which:

   i. $f(x)$ is a decreasing function.
   ii. $f(x) \,.\, g(x) \geq 0$.
   iii. $f'(x)$ and $g(x)$ are both negative.

   Solution:

   i. $x < -1$ and $x > 3$
   ii. $x = -3$ or $x \geq 3$ (both f and g are negative)
   iii. $f'(x) < 0$ where the gradient of f is negative. Therefore, $x < -1$ or $x > 3$.
   And $g < 0$ for $x < -3$ or $x > 3$.
   Therefore, f' and $g < 0$ for $x < -3$ or $x > 3$.

b) Given $f(x) = -x^3 + 3x^2 + 9x - 27$, determine the equation of the tangent to f at the point $E(2; y)$.

Solution:

Equation of tangent to f at $E(2;y)$:

$$\begin{aligned} f'(x) &= -3x^2 + 6x + 9 \\ \text{At } x = 2: \quad f'(2) &= -3(2)^2 + 6(2) + 9 \\ &= -12 + 12 + 9 \\ &= 9 \\ \therefore y &= 9x + c \end{aligned}$$

To calculate the value of y at E, substitute $x = 2$ into $f(x)$:

$$\begin{aligned} f(2) &= -(2)^3 + 3(2)^2 + 9(2) - 27 \\ &= -8 + 12 + 18 - 27 \\ &= -5 \end{aligned}$$

Substitute $(2;-5)$ into $f'(x)$ to determine c:

$$\begin{aligned} y &= 9x + c \\ -5 &= 9(2) + c \\ \therefore c &= -23 \\ y &= 9x - 23 \end{aligned}$$

c) Find the coordinates of the point(s) where the tangent in the question above meets the graph of f again.

Solution:

Equate the equation of the tangent and $f(x)$:

$$\begin{aligned} 9x - 23 &= -x^3 + 3x^2 + 9x - 27 \\ \therefore 0 &= -x^3 + 3x^2 - 4 \\ \text{Let } k(x) &= -x^3 + 3x^2 - 4 \\ k(-1) &= -(-1)^3 + 3(-1)^2 - 4 \\ &= 1 + 3 - 4 \\ &= 0 \\ \therefore k(x) &= (x+1)(-x^2 + 4x - 4) \\ &= -(x+1)(x^2 - 4x + 4) \\ &= -(x+1)(x-2)^2 \\ \therefore 0 &= -(x+1)(x-2)^2 \\ \therefore x = -1 &\text{ or } x = 2 \text{ or } x = 2 \end{aligned}$$

Therefore, tangent meets graph of f at the turning point $C(-1;-32)$.

d) Without any calculations, give the x-intercepts of the graph of $f'(x)$. Explain reasoning.

Solution:

x-intercepts of f' are at $(-1;0)$ and $(3;0)$, the turning points of f (points B and C). At these two points the gradient of the graph is equal to zero, where $f'(x)$ cuts the x-axis.

17. a) Sketch the graph of $f(x) = x^3 - 9x^2 + 24x - 20$, show all intercepts with the axes and turning points.

Solution:

We find the y-intercept by finding the value for $f(0)$.

$$\begin{aligned} f(x) &= x^3 - 9x^2 + 24x - 20 \\ f(0) &= (0)^3 - 9(0)^2 + 24(0) - 20 \\ &= -20 \end{aligned}$$

The y-intercept is: $(0; -20)$
We find the x-intercepts by finding the values for which the function $f(x) = 0$.
We use the factor theorem to check if $(x - 1)$ is a factor.

$$\begin{aligned} f(x) &= x^3 - 9x^2 + 24x - 20 \\ f(1) &= (1)^3 - 9(1)^2 + 24(1) - 20 \\ &= -4 \end{aligned}$$

Therefore, $(x - 1)$ is not a factor.
We now use the factor theorem to check if $(x + 1)$ is a factor.

$$\begin{aligned} f(x) &= x^3 - 9x^2 + 24x - 20 \\ f(-1) &= (-1)^3 - 9(-1)^2 + 24(-1) - 20 \\ &= -54 \end{aligned}$$

Therefore, $(x + 1)$ is not a factor.
We now try $(x - 2)$:

$$\begin{aligned} f(x) &= x^3 - 9x^2 + 24x - 20 \\ f(1) &= (2)^3 - 9(2)^2 + 24(2) - 20 \\ &= 0 \end{aligned}$$

Therefore, $(x - 2)$ is a factor.
If we divide $f(x)$ by $(x - 2)$ we are left with:

$$f(x) = (x - 2)(x^2 - 7x + 10)$$

This has factors:

$$f(x) = (x - 2)(x - 5)(x - 2)$$

The x-intercepts are: $(2; 0), (5; 0)$.
Find the turning points by setting $f'(x) = 0$.
If we use the rules of differentiation we get:

$$\begin{aligned} f'(x) &= 3x^2 - 18x + 24 \\ 0 &= 3(x^2 - 6x + 8) \\ &= 3(x - 2)(x - 4) \end{aligned}$$

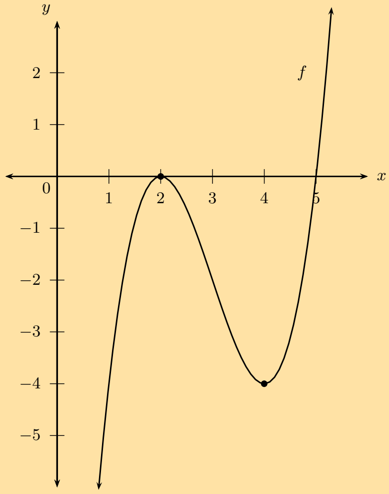
The x-coordinates of the turning points are: $x = 4$ and $x = 2$.
The y-coordinates of the turning points are calculated as:

$$\begin{aligned} f(2) &= (2)^3 - 9(2)^2 + 24(2) - 20 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} f(4) &= (4)^3 - 9(4)^2 + 24(4) - 20 \\ &= -4 \end{aligned}$$

Therefore the turning points are: $(2; 0)$ and $(4; -4)$.



b) Find the equation of the tangent to $f(x)$ at $x = 4$.

Solution:

We know that at $x = 4$, $y = -4$ (this is a turning point of the graph).

We substitute $x = 4$ into the derivative of the function to get m:

$$m = 3(4-2)(4-4)$$
$$m = 0$$

Substituting this and the coordinates of the point into: $y - y_1 = m(x - x_1)$ gives:

$$y - (-4) = 0(x-4)$$
$$y = -4$$

The equation of the tangent to the graph at $x = 4$ is $y = -4$.

c) Determine the point of inflection and discuss the concavity of f.

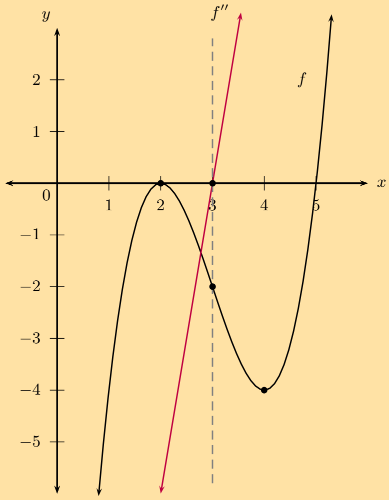
Solution:

Point of inflection:

$$f'(x) = 3x^2 - 18x + 24$$
$$f''(x) = 6x - 18$$
$$\therefore 0 = 6x - 18$$
$$6x = 18$$
$$\therefore x = 3$$

$$\text{Substitute } x = 3: \quad f(3) = (3)^3 - 9(3)^2 + 24(3) - 20$$
$$= 27 - 81 + 72 - 20$$
$$= -2$$

This gives the point $(3; -2)$.



Concavity:
$f''(x) < 0$ for $x < 3$: concave down
$f''(x) = 0$ for $x = 3$: point of inflection
$f''(x) > 0$ for $x > 3$: concave up

18. Determine the minimum value of the sum of a positive number and its reciprocal.

Solution:

Let the number be x and the reciprocal be $\frac{1}{x}$. The sum of these two numbers is $S = x + \frac{1}{x}$. To find the minimum value we must differentiate the expression for the sum and set it equal to zero:

$$\begin{aligned} S &= x + \frac{1}{x} \\ S' &= 1 - \frac{1}{x^2} \\ 0 &= 1 - \frac{1}{x^2} \\ 1 &= \frac{1}{x^2} \\ x^2 &= 1 \\ x &= 1 \\ \therefore S &= 1 + \frac{1}{1} \\ &= 2 \end{aligned}$$

Therefore, the minimum value for the sum of a positive number and it's reciprocal is 2.

19. t minutes after a kettle starts to boil, the height of the water in the kettle is given by $d = 86 - \frac{1}{8}t - \frac{1}{4}t^3$, where d is measured in millimetres.

a) Calculate the height of the water level in the kettle just before it starts to boil.

Solution:

The kettle starts to boil at $t = 0$:

$$\begin{aligned} d &= 86 - \frac{1}{8}t - \frac{1}{4}t^3 \\ &= 86 - \frac{1}{8}(0) - \frac{1}{4}(0)^3 \\ &= 86 \text{ mm} \end{aligned}$$

b) As the water boils, the water level in the kettle decreases. Determine the rate at which the water level is decreasing when $t = 2$ minutes.

Solution:

To find the rate at which the water level is decreasing we take the derivative of the depth:

$$\begin{aligned} d(t) &= 86 - \frac{1}{8}t - \frac{1}{4}t^3 \\ d'(t) &= -\frac{1}{8} - \frac{1}{4}(3)t^2 \\ &= -\frac{1}{8} - \frac{3}{4}t^2 \\ d'(2) &= -\frac{1}{8} - \frac{3}{4}(2)^2 \\ &= -\frac{1}{8} - 3 \\ &= -\frac{25}{8} \\ &= -3{,}125 \text{ mm per minute} \end{aligned}$$

This rate is negative since the water level is decreasing.

c) How many minutes after the kettle starts to boil will the water level be decreasing at a rate of $12\frac{1}{8}$ mm per minute?

Solution:

To find the number of minutes when the rate will be $-12{,}125$ mm.min^{-1} (water level is decreasing) we set the derivative equal to this and solve for t:

$$\begin{aligned} d'(t) &= -\frac{1}{8} - \frac{3}{4}t^2 \\ -12{,}125 &= -\frac{1}{8} - \frac{3}{4}t^2 \\ \frac{97}{8} - \frac{1}{8} &= \frac{3}{4}t^2 \\ \frac{3}{4}t^2 &= 12 \\ t^2 &= 16 \\ t &= 4 \end{aligned}$$

After 4 minutes the water level will be decreasing at $12{,}125$ mm.min^{-1}.

20. The displacement of a moving object is represented by the equation:

$$\begin{aligned} D(t) &= \frac{4}{3}t^3 - 3t \\ \text{where } D &= \text{ distance travelled in metres} \\ t &= \text{ time in seconds} \end{aligned}$$

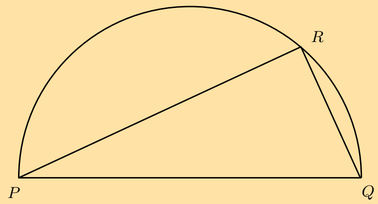
Calculate the acceleration of the object after 3 seconds.

Solution:

$$\begin{aligned} D(t) &= \frac{4}{3}t^3 - 3t \\ D'(t) &= 4t^2 - 3 \\ D''(t) &= 8t \\ \therefore D''(3) &= 8(3) \\ &= 24 \end{aligned}$$

After 3 seconds, the acceleration is 24 m.s^{-2}.

21. In the figure PQ is the diameter of the semi-circle PRQ. The sum of the lengths of PR and QR is 10 units. Calculate the perimeter of $\triangle PQR$ when $\triangle PQR$ covers the maximum area in the semi-circle. Leave the answer in simplified surd form.



Solution:

Let PR be x units and QR be $(10 - x)$ units.

$$\begin{aligned} \text{Area } \triangle PQR &= \frac{1}{2}PR\ .\ QR \quad (\angle \text{ in semi-circle}) \\ \therefore A(x) &= \frac{1}{2}x(10 - x) \\ &= 5x - \frac{1}{2}x^2 \end{aligned}$$

To find maximum area, let $A'(x) = 0$:

$$
\begin{aligned}
A'(x) &= 5 - x \\
0 &= 5 - x \\
\therefore x &= 5
\end{aligned}
$$

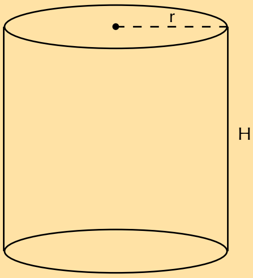
Therefore, $PR = 5$ units and $QR = 5$ units.

$$
\begin{aligned}
PQ^2 &= 5^2 + 5^2 \quad \text{(Pythagoras)} \\
&= 50 \\
\therefore PQ &= \sqrt{50} \\
&= 5\sqrt{2}
\end{aligned}
$$

$$
\begin{aligned}
\text{Perimeter } \triangle PQR &= 5 + 5 + 5\sqrt{2} \\
&= 5 + 5 + 5\sqrt{2} \\
&= 10 + 5\sqrt{2} \text{ units}
\end{aligned}
$$

Perimeter of the triangle is $10 + 5\sqrt{2}$ units.

22. The capacity of a cylindrical water tank is 1000 litres. Let the height be H and the radius be r. The material used for the bottom of the tank is twice as thick and also twice as expensive as the material used for the curved part of the tank and the top of the tank.
Remember: $1000\ \ell = 1\ \text{m}^3$



a) Express H in terms of r.
Solution:

$$
\begin{aligned}
V &= \pi r^2 H \\
1 &= \pi r^2 H \\
\therefore H &= \frac{1}{\pi r^2}
\end{aligned}
$$

b) Show that the cost of the material for the tank can be expressed as:

$$C = 3\pi r^2 + \frac{2}{r}$$

Solution:

$$
\begin{aligned}
\text{Cost of material} &= (2 \times \text{area bottom}) + \text{area top} + \text{curved part} \\
&= \left(2 \times \pi r^2\right) + \pi r^2 + (2\pi r H) \\
&= 3\pi r^2 + 2\pi r H \\
\text{Substitute } H &= \frac{1}{\pi r^2}: \\
\text{Cost of material} &= 3\pi r^2 + \left(2\pi r \times \frac{1}{\pi r^2}\right) \\
&= 3\pi r^2 + \frac{2}{r}
\end{aligned}
$$

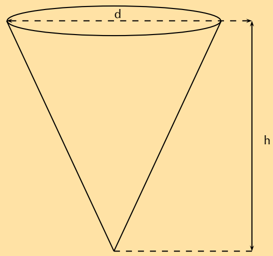
c) Determine the diameter of the tank that gives the minimum cost of the materials. [IEB, 2006]

Solution:

Let the cost of the material be $C(r)$.

$$\begin{aligned} C(r) &= 3\pi r^2 + \frac{2}{r} \\ C'(r) &= 6\pi r - \frac{2}{r^2} \\ \therefore 0 &= 6\pi r - \frac{2}{r^2} \\ r^3 &= \frac{2}{6\pi} \\ \therefore r &= 0{,}47 \text{ m} \\ \therefore d &= 2 \times 0{,}47 \text{ m} \\ &= 0{,}94 \text{ m} \end{aligned}$$

23. The diameter of an icecream cone is d and the vertical height is h. The sum of the diameter and the height of the cone is 10 cm.



a) Determine the volume of the cone in terms of h and d.
(Volume of a cone: $V = \frac{1}{3}\pi r^3 h$)

Solution:

$$\begin{aligned} r &= \frac{d}{2} \\ h + d &= 10 \\ \therefore h &= 10 - d \\ V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi \left(\frac{d}{2}\right)^2 (10 - d) \\ &= \frac{1}{12}\pi d^2(10 - d) \\ &= \frac{1}{12}(10\pi d^2 - \pi d^3) \end{aligned}$$

b) Determine the radius and height of the cone for the volume to be a maximum.

Solution:

Let the volume of the cone be $V(d)$.

$$\begin{aligned} V(d) &= \frac{1}{12}(10\pi d^2 - \pi d^3) \\ V'(d) &= \frac{1}{12}(20\pi d - 3\pi d^2) \\ \therefore 0 &= \frac{1}{12}(20\pi d - 3\pi d^2) \\ 0 &= 20\pi d - 3\pi d^2 \\ 0 &= \pi d(20 - 3d) \\ \therefore d &= 0 \text{ or } d = \frac{20}{3} \end{aligned}$$

$$\begin{aligned}
\therefore d &= 6{,}67 \text{ cm} \\
\therefore r &= \frac{1}{2} \times \frac{20}{3} \\
&= \frac{10}{3} \\
&= 3{,}34 \text{ cm} \\
h &= 10 - \frac{20}{3} \\
&= \frac{10}{3} \\
&= 3{,}34 \text{ cm}
\end{aligned}$$

c) Calculate the maximum volume of the cone.

Solution:

$$\begin{aligned}
V &= \frac{1}{3}\pi r^2 h \\
&= \frac{1}{3}\pi \left(\frac{10}{3}\right)^2 \left(\frac{10}{3}\right) \\
&= \frac{1}{3}\pi \left(\frac{100}{9}\right) \left(\frac{10}{3}\right) \\
&= \frac{1000\pi}{81} \\
&= 38{,}79 \text{ cm}^3
\end{aligned}$$

24. A water reservoir has both an inlet and an outlet pipe to regulate the depth of the water in the reservoir. The depth is given by the function:

$$\begin{aligned}
D(h) &= 3 + \frac{1}{2}h - \frac{1}{4}h^3 \\
\text{where } D &= \text{ depth in metres} \\
h &= \text{ hours after 06h00}
\end{aligned}$$

a) Determine the rate at which the depth of the water is changing at 10h00.

Solution:

$$\begin{aligned}
D(h) &= 3 + \frac{1}{2}h - \frac{1}{4}h^3 \\
D'(h) &= \frac{1}{2} - \frac{3}{4}h^2 \\
\therefore D'(4) &= \frac{1}{2} - \frac{3}{4}(4)^2 \\
&= \frac{1}{2} - 12 \\
&= -11\frac{1}{2}
\end{aligned}$$

Therefore, rate of change of the depth of the water is $-11{,}5$ m per hour at 10h00.

b) Is the depth of the water increasing or decreasing?

Solution:

Decreasing (rate is negative).

c) At what time will the inflow of water be the same as the outflow?

[IEB, 2006]

Solution:

$$D'(h) = \frac{1}{2} - \frac{3}{4}h^2$$
$$0 = \frac{1}{2} - \frac{3}{4}h^2$$
$$\frac{3}{4}h^2 = \frac{1}{2}$$
$$h^2 = \frac{2}{3}$$
$$\therefore h = 0{,}82 \text{ hours}$$

We can convert this to minutes: $0{,}82 \times 60 \approx 49$ minutes. Therefore, at 06h49

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28ZC	2. 28ZD	3. 28ZF	4a. 28ZG	4b. 28ZH	4c. 28ZJ
4d. 28ZK	5a. 28ZM	5b. 28ZN	6. 28ZP	7a. 28ZQ	7b. 28ZR
8a. 28ZS	8b. 28ZT	9. 28ZV	10a. 28ZW	10b. 28ZX	10c. 28ZY
11a. 28ZZ	11b. 2922	11c. 2923	11d. 2924	11e. 2925	11f. 2926
11g. 2927	12a. 2928	12b. 2929	12c. 292B	12d. 292C	12e. 292D
13a. 292F	13b. 292G	13c. 292H	13d. 292J	13e. 292K	14a. 292M
14b. 292N	14c. 292P	15. 292Q	16a. 292R	16b. 292S	16c. 292T
16d. 292V	17. 292W	18. 292X	19a. 292Y	19b. 292Z	19c. 2932
20. 2933	21. 2934	22a. 2935	22b. 2936	22c. 2937	23a. 2938
23b. 2939	23c. 293B	24a. 293C	24b. 293D	24c. 293F	

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CHAPTER 7

Analytical geometry

7.1 Revision

- Integrate Euclidean Geometry knowledge with Analytical Geometry.
- Emphasize the value and importance of making sketches.
- Emphasize the importance of writing coordinates consistently for the distance formula and gradient.
- Learners must revise the method of completing the square for finding the general form of the equation of a circle with centre $(a; b)$.
- Remind learners that the tangent to a circle is perpendicular to the radius (and the diameter).

Straight line equations

Exercise 7 – 1: Revision

1. Determine the following for the line segment between the given points:
   - length
   - mid-point
   - gradient
   - equation

   a) $(-2; -4)$ and $(3; 11)$
   Solution:

$$\begin{aligned} \text{Distance} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - (-2))^2 + (11 - (-4))^2} \\ &= \sqrt{(5)^2 + (15)^2} \\ &= \sqrt{25 + 225} \\ &= \sqrt{250} \\ &= 5\sqrt{10} \end{aligned}$$

$$\begin{aligned} M(x; y) &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{-2 + 3}{2}; \frac{-4 + 11}{2}\right) \\ &= \left(\frac{1}{2}; \frac{7}{2}\right) \end{aligned}$$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{11 - (-4)}{3 - (-2)} \\ &= \frac{15}{5} \\ &= 3 \end{aligned}$$

$$
\begin{aligned}
y &= mx + c \\
y &= 3x + c \\
\text{Substitute } (-2;-4) \quad -4 &= 3(-2) + c \\
-4 + 6 &= c \\
2 &= c \\
\therefore y &= 3x + 2
\end{aligned}
$$

b) $(-5;-3)$ and $(10;6)$

Solution:

$$
\begin{aligned}
\text{Distance } &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(10 - (-5))^2 + (6 - (-3))^2} \\
&= \sqrt{(15)^2 + (9)^2} \\
&= \sqrt{225 + 81} \\
&= \sqrt{306}
\end{aligned}
$$

$$
\begin{aligned}
M(x;y) &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\
&= \left(\frac{-5 + 10}{2}; \frac{-3 + 6}{2}\right) \\
&= \left(\frac{5}{2}; \frac{3}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{6 - (-3)}{10 - (-5)} \\
&= \frac{9}{15} \\
&= \frac{3}{5}
\end{aligned}
$$

$$
\begin{aligned}
y &= mx + c \\
y &= \frac{3}{5}x + c \\
\text{Substitute } (10;6) \quad 6 &= \frac{3}{5}(10) + c \\
6 - 6 &= c \\
0 &= c \\
\therefore y &= \frac{3}{5}x
\end{aligned}
$$

c) $(h;-h-k)$ and $(2k;h-5k)$

Solution:

$$
\begin{aligned}
\text{Distance } &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(h - 2k)^2 + (-h - k - (h - 5k))^2} \\
&= \sqrt{(h - 2k)^2 + (-2h + 4k)^2} \\
&= \sqrt{h^2 - 4hk + 4k^2 + +4h^2 - 16hk + 16k^2} \\
&= \sqrt{5h^2 - 20hk + 20k^2}
\end{aligned}
$$

$$\begin{aligned} M(x;y) &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right) \\ &= \left(\frac{h+2k}{2};\frac{-h-k+(h-5k)}{2}\right) \\ &= \left(\frac{h+2k}{2};\frac{-6k)}{2}\right) \\ &= \left(\frac{h+2k}{2};-3k\right) \end{aligned}$$

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{(h-5k)-(-h-k)}{2k-h} \\ &= \frac{2h-4k}{-h+2k} \\ &= \frac{-2(-h+2k)}{-h+2k} \\ &= -2 \end{aligned}$$

$$\begin{aligned} y &= mx+c \\ y &= -2x+c \\ \text{Substitute } (h;-h-k) \quad -h-k &= -2(h)+c \\ -h+2h-k &= c \\ h-k &= c \\ \therefore y &= -2x+h-k \end{aligned}$$

d) $(2;9)$ and $(0;-1)$

Solution:

$$\begin{aligned} \text{Distance } &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ &= \sqrt{(2-(0))^2+(9-(-1))^2} \\ &= \sqrt{(2)^2+(10)^2} \\ &= \sqrt{4+100} \\ &= \sqrt{104} \end{aligned}$$

$$\begin{aligned} M(x;y) &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right) \\ &= \left(\frac{2+0}{2};\frac{9-1}{2}\right) \\ &= (1;4) \end{aligned}$$

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{-1-9}{0-2} \\ &= \frac{-10}{-2} \\ &= 5 \end{aligned}$$

$$\begin{aligned} y &= mx+c \\ y &= 5x+c \\ \text{Substitute } (0;-1) \quad -1 &= 5(0)+c \\ -1 &= c \\ \therefore y &= 5x-1 \end{aligned}$$

2. The line joining $A(x;y)$ and $B(-3;6)$ has the mid-point $M(2;3)$. Determine the values of x and y.

Solution:

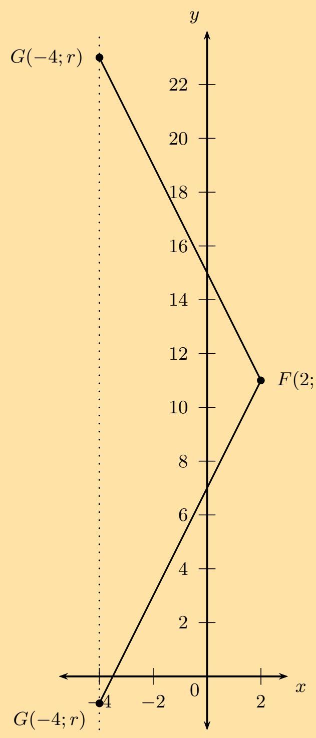
$$M(x;y) = \left(\frac{x_A + x_B}{2}; \frac{y_A + y_B}{2}\right)$$
$$M(2;3) = \left(\frac{x-3}{2}; \frac{y+6}{2}\right)$$
$$\therefore 2 = \frac{x-3}{2}$$
$$4 = x - 3$$
$$\therefore 7 = x$$
$$\text{And } 3 = \frac{y+6}{2}$$
$$6 = y + 6$$
$$\therefore 0 = y$$

$A(7;0)$

3. Given $F(2;11), G(-4;r)$ and length $FG = 6\sqrt{5}$ units, determine the value(s) of r.

Solution:

There are two possible values of r such that the length $FG = 6\sqrt{5}$ units:



$$FG = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$6\sqrt{5} = \sqrt{(-4-2)^2 + (r-11)^2}$$
$$\left(6\sqrt{5}\right)^2 = 36 + r^2 - 22r + 121$$
$$36 \times 5 = r^2 - 22r + 157$$
$$0 = r^2 - 22r - 23$$
$$0 = (r+1)(r-23)$$
$$\therefore r = -1 \text{ or } r = 23$$

4. Determine the equation of the straight line:

a) passing through the point $(\frac{1}{2};4)$ and $(1;5)$.
Solution:

$$\begin{aligned}
\frac{y-y_1}{x-x_2} &= \frac{y_2-y_1}{x_2-x_1} \\
\frac{y-4}{x-\frac{1}{2}} &= \frac{5-4}{1-\frac{1}{2}} \\
\frac{y-4}{x-\frac{1}{2}} &= \frac{1}{\frac{1}{2}} \\
\frac{y-4}{x-\frac{1}{2}} &= 2 \\
y-4 &= 2\left(x-\frac{1}{2}\right) \\
y &= 2x-1+4 \\
\therefore y &= 2x+3
\end{aligned}$$

b) passing through the points $(2;-3)$ and $(-1;0)$.
Solution:

$$\begin{aligned}
y &= mx+c \\
-3 &= 2m+c\ldots(1) \\
0 &= -m+c\ldots(2) \\
(1)-(2):\quad -3 &= 2m+m \\
-3 &= 3m \\
\therefore -1 &= m \\
\therefore c &= -1 \\
\therefore y &= -x-1
\end{aligned}$$

c) passing through the point $(9;1)$ and with $m=\frac{1}{3}$.
Solution:

$$\begin{aligned}
y-y_1 &= m(x-x_1) \\
y-1 &= \frac{1}{3}(x-9) \\
y-1 &= \frac{1}{3}x-3 \\
\therefore y &= \frac{1}{3}x-2
\end{aligned}$$

d) parallel to the x-axis and passing through the point $(0;-4)$.
Solution:

$$\begin{aligned}
y-y_1 &= m(x-x_1) \\
y-(-4) &= 0(x-0) \\
\therefore y &= -4
\end{aligned}$$

e) passing through the point $(\frac{1}{2};-1)$ and with $m=-4$.
Solution:

$$\begin{aligned}
y-y_1 &= m(x-x_1) \\
y-(-1) &= -4(x-\frac{1}{2}) \\
y+1 &= -4x+2 \\
\therefore y &= -4x+1
\end{aligned}$$

f) perpendicular to the x-axis and passing through the point $(5; 0)$.
Solution: $x = 5$

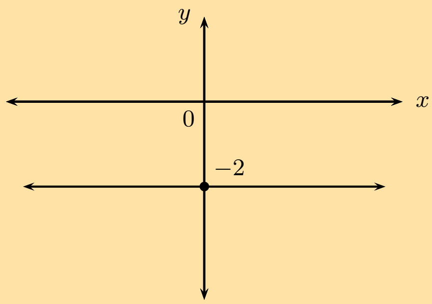
g) with undefined gradient and passing through the point $(\frac{3}{4}; 0)$.
Solution: $x = \frac{3}{4}$

h) with $m = 2p$ and passing through the point $(3; 6p + 3)$.
Solution:

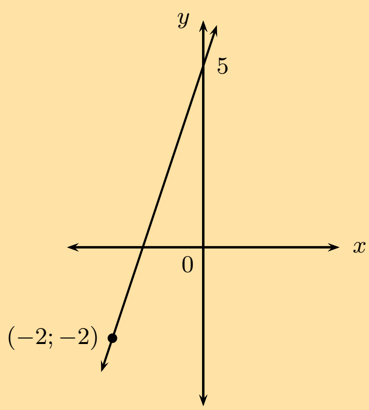
$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (6p + 3) &= 2p(x - 3) \\ y - 6p - 3 &= 2px - 6p \\ \therefore y &= 2px + 3 \end{aligned}$$

i) which cuts the y-axis at $y = -\frac{3}{5}$ and with $m = 4$.
Solution:

$$\begin{aligned} y &= mx + c \\ y &= mx - \frac{3}{5} \\ \therefore y &= 4x - \frac{3}{5} \end{aligned}$$

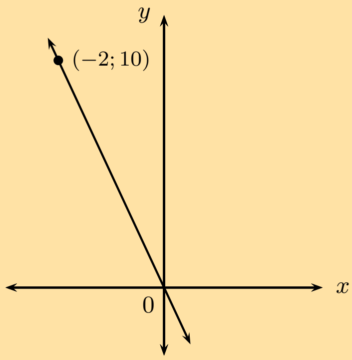
j) 

Solution: $y = -2$

k) 

Solution:

$$\begin{aligned} c &= 5 \\ y &= mx + c \\ y &= mx + 5 \\ \text{Substitute } (-2; -2) \quad -2 &= -2m + 5 \\ -7 &= -2m \\ m &= \frac{7}{2} \\ \therefore y &= \frac{7}{2}x + 5 \end{aligned}$$



l)

Solution:

$$\begin{aligned} c &= 0 \\ y &= mx + c \\ y &= mx + 0 \\ \text{Substitute } (-2;10) \quad 10 &= -2m \\ -2m &= 10 \\ \therefore m &= -5 \\ \therefore y &= -5x \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 293J	1b. 293K	1c. 293M	1d. 293N	2. 293P	3. 293Q
4a. 293R	4b. 293S	4c. 293T	4d. 293V	4e. 293W	4f. 293X
4g. 293Y	4h. 293Z	4i. 2942	4j. 2943	4k. 2944	4l. 2945

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Inclination of a line

Exercise 7 – 2: Inclination of a straight line

1. Determine the angle of inclination (correct to 1 decimal place) for each of the following:

   a) a line with $m = \frac{3}{4}$

   Solution:

$$\begin{aligned} \tan\theta &= m \\ &= \frac{3}{4} \\ \theta &= \tan^{-1}(0{,}75) \\ \therefore \theta &= 36{,}9^\circ \end{aligned}$$

   b) $6 + x = 2y$

   Solution:

$$\begin{aligned} 6 + x &= 2y \\ 2y &= x + 6 \\ y &= \frac{1}{2}x + 3 \\ \tan\theta &= m \\ &= \frac{1}{2} \\ \theta &= \tan^{-1}(0{,}5) \\ \therefore \theta &= 26{,}6^{\circ} \end{aligned}$$

c) the line passes through the points $(-4; 0)$ and $(2; 6)$

Solution:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{6 - 0}{2 - (-4)} \\ &= \frac{6}{6} \\ \therefore m &= 1 \\ \tan\theta &= 1 \\ \theta &= \tan^{-1}(1) \\ \therefore \theta &= 45^{\circ} \end{aligned}$$

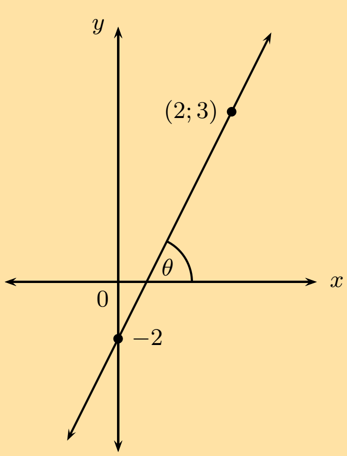
d) $y = 4$

Solution: Horizontal line

e) a line with a gradient of $1{,}733$

Solution:

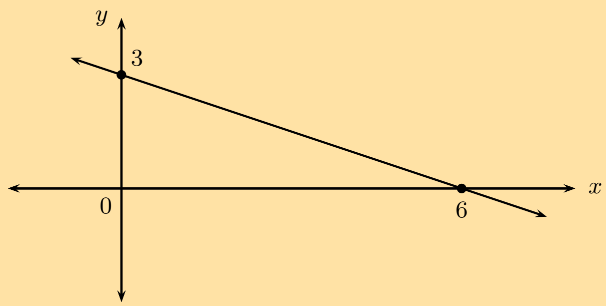
$$\begin{aligned} m &= 1{,}733 \\ \theta &= \tan^{-1}(1{,}733) \\ \therefore \theta &= 60^{\circ} \end{aligned}$$



f)

Solution:

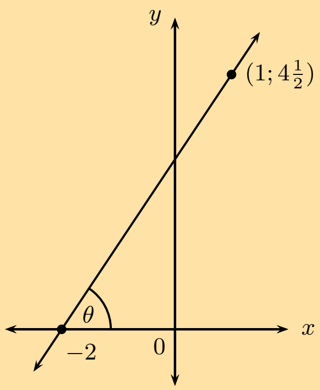
$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{3+2}{2-0} \\
&= \frac{5}{2} \\
\theta &= \tan^{-1}\left(\frac{5}{2}\right) \\
\therefore \theta &= 68{,}2^\circ
\end{aligned}
$$



g)

Solution:

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{3-0}{0-6} \\
&= \frac{3}{-6} \\
\therefore m &= -\frac{1}{2} \\
\theta &= \tan^{-1}\left(-\frac{1}{2}\right) \\
\therefore \theta &= -26{,}6^\circ \\
\therefore \theta &= 180^\circ - 26{,}6^\circ \\
\therefore \theta &= 153{,}4^\circ
\end{aligned}
$$



h)

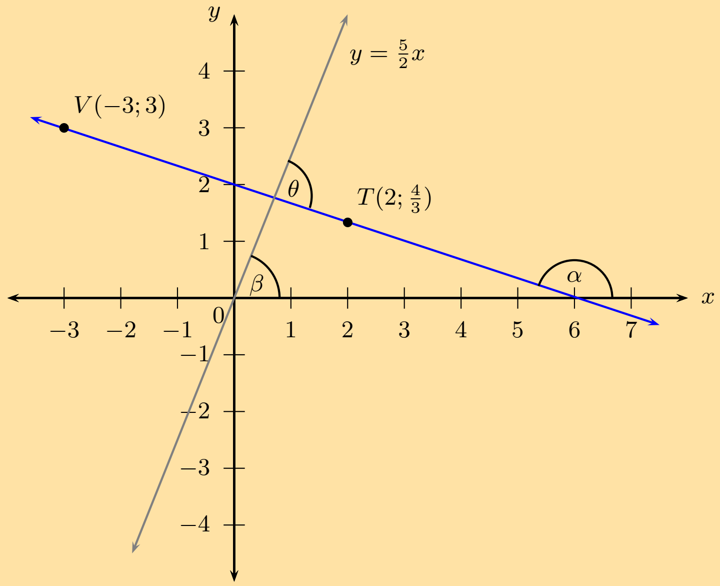
Solution:

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{\frac{9}{2} - 0}{1 + 2} \\
&= \frac{\frac{9}{2}}{3} \\
&= \frac{3}{2} \\
\theta &= \tan^{-1}\left(\frac{3}{2}\right) \\
\therefore \theta &= 56{,}3^{\circ}
\end{aligned}
$$

2. Find the angle between the line $2y = 5x$ and the line passing through points $T(2; 1\frac{1}{3})$ and V$(-3; 3)$.

Solution:

Let the angle of inclination for the line $2y = 5x$ be β and let the angle of inclination for the other line be α. Let the angle between the two lines be θ.



$$
\begin{aligned}
2y &= 5x \\
y &= \frac{5}{2}x \\
\therefore m &= \frac{5}{2} \\
\beta &= \tan^{-1}\left(\frac{5}{2}\right) \\
\therefore \beta &= 68{,}2^{\circ} \\
m_{TV} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{3 - \frac{4}{3}}{-3 - 2} \\
&= \frac{\frac{5}{3}}{-5} \\
\therefore m_{TV} &= -\frac{1}{3} \\
m_{TV} = \tan\alpha &= -\frac{1}{3} \\
\alpha &= \tan^{-1}\left(-\frac{1}{3}\right) \\
&= -18{,}4^{\circ} \\
\alpha &= 180^{\circ} - 18{,}4^{\circ} \\
\therefore \alpha &= 161{,}6^{\circ} \\
\text{And } \theta &= \beta + (180^{\circ} - \alpha) \quad (\text{ext. } \angle\triangle) \\
\therefore \theta &= 68{,}2^{\circ} + (180^{\circ} - 161{,}6^{\circ}) \\
&= 86{,}6^{\circ}
\end{aligned}
$$

3. Determine the equation of the straight line that passes through the point $(1;2)$ and is parallel to the line $y + 3x = 1$.

Solution:

$$
\begin{aligned}
y + 3x &= 1 \\
y &= -3x + 1 \\
\therefore m &= -3 \\
y - y_1 &= m(x - x_1) \\
y - 2 &= -3(x - 1) \\
y &= -3x + 3 + 2 \\
\therefore y &= -3x + 5
\end{aligned}
$$

4. Determine the equation of the straight line that passes through the point $(-4;-4)$ and is parallel to the line with angle of inclination $\theta = 56{,}31^{\circ}$.

Solution:

$$
\begin{aligned}
\theta &= 56{,}31^{\circ} \\
\therefore m &= \tan\theta \\
&= \tan 56{,}31^{\circ} \\
\therefore m &= 1{,}5 \\
y - y_1 &= m(x - x_1) \\
y + 4 &= \frac{3}{2}(x + 4) \\
y &= \frac{3}{2}x + 6 - 4 \\
\therefore y &= \frac{3}{2}x + 2
\end{aligned}
$$

5. Determine the equation of the straight line that passes through the point $(1;-6)$ and is perpendicular to the line $5y = x$.

Solution:

$$\begin{aligned} 5y &= x \\ y &= \frac{1}{5}x \\ \therefore m_1 &= \frac{1}{5} \\ \text{For } \perp\text{: } m_1 \times m_2 &= -1 \\ \frac{1}{5} \times m_2 &= -1 \\ \therefore m_2 &= -5 \\ y &= mx + c \\ y &= -5x + c \\ \text{Substitute } (1;-6)\text{ : } \quad -6 &= -5(1) + c \\ -6 &= -5 + c \\ \therefore c &= -1 \\ \therefore y &= -5x - 1 \end{aligned}$$

6. Determine the equation of the straight line that passes through the point $(3;-1)$ and is perpendicular to the line with angle of inclination $\theta = 135°$.

Solution:

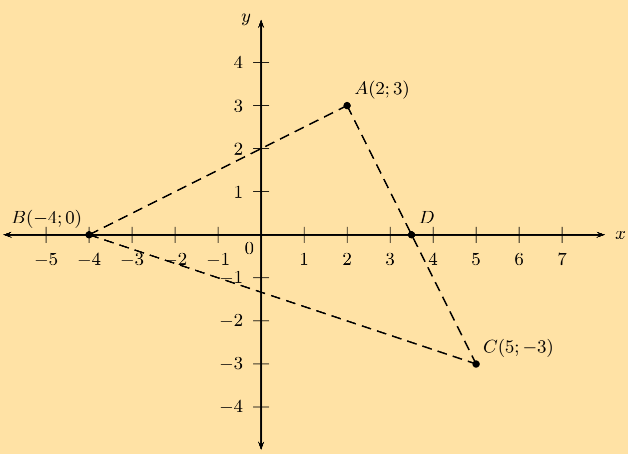
$$\begin{aligned} \theta &= 135° \\ \therefore m_1 &= \tan\theta \\ &= \tan 135° \\ \therefore m_1 &= -1 \\ m_1 \times m_2 &= -1 \\ \therefore m_2 &= 1 \\ y &= mx + c \\ y &= x + c \\ \text{Substitute } (3;-1)\text{ : } \quad -1 &= (3) + c \\ \therefore c &= -4 \\ \therefore y &= x - 4 \end{aligned}$$

7. $A(2;3)$, $B(-4;0)$ and $C(5;-3)$ are the vertices of $\triangle ABC$ in the Cartesian plane. AC intersects the x-axis at D. Draw a sketch and determine the following:

a) the equation of line AC

Solution:

Draw a sketch:



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-3 - 3}{5 - 2}$$
$$= \frac{-6}{3}$$
$$\therefore m = -2$$
$$y = mx + c$$
$$\therefore y = -2x + c$$
$$\text{Substitute } (2;3): \quad 3 = -2(2) + c$$
$$\therefore c = 7$$
$$\therefore y = -2x + 7$$

b) the coordinates of point D

Solution:

$$y = -2x + 7$$
$$0 = -2x + 7$$
$$\therefore x = \frac{7}{2}$$
$$\therefore D\left(\frac{7}{2};0\right)$$

c) the angle of inclination of AC

Solution:

$$\therefore m = -2$$
$$\tan\theta = m$$
$$\tan\theta = -2$$
$$\therefore \theta = \tan^{-1}(-2)$$
$$\theta = -63{,}4^\circ + 180^\circ$$
$$\therefore \theta = 116{,}6^\circ$$

d) the gradient of line AB

Solution:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{3 - 0}{2 + 4}$$
$$= \frac{3}{6}$$
$$\therefore m = \frac{1}{2}$$

e) $B\hat{A}C$

Solution:

$B\hat{A}C = 90^\circ$ because $m_{AB} \times m_{AC} = -1$

f) the equation of the line perpendicular to AB and passing through the origin

Solution:

$$m_{PQ} = \frac{1}{2}$$
$$\therefore m_{\perp} = -2$$
$$y = -2x + c$$
$$c = 0$$
$$\therefore y = -2x$$

g) the mid-point M of BC

Solution:

$$\begin{aligned} M(x;y) &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right) \\ &= \left(\frac{-4+5}{2};\frac{0-3}{2}\right) \\ &= \left(\frac{1}{2};-\frac{3}{2}\right) \end{aligned}$$

h) the equation of the line parallel to AC and passing through point M

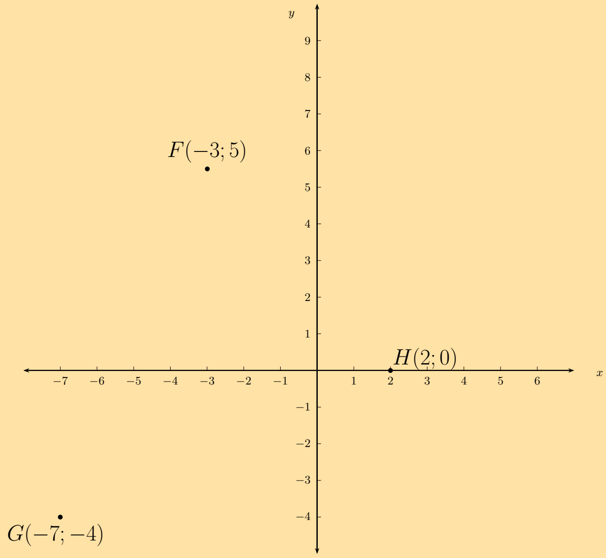
Solution:

$$\begin{aligned} m &= -2 \\ y &= mx+c \\ y &= -2x+c \\ \text{Substitute } \left(\frac{1}{2};-\frac{3}{2}\right): \quad -\frac{3}{2} &= -2\left(\frac{1}{2}\right)+c \\ c &= -\frac{1}{2} \\ y &= -2x-\frac{1}{2} \end{aligned}$$

8. Points $F(-3;5)$, $G(-7;-4)$ and $H(2;0)$ are given.

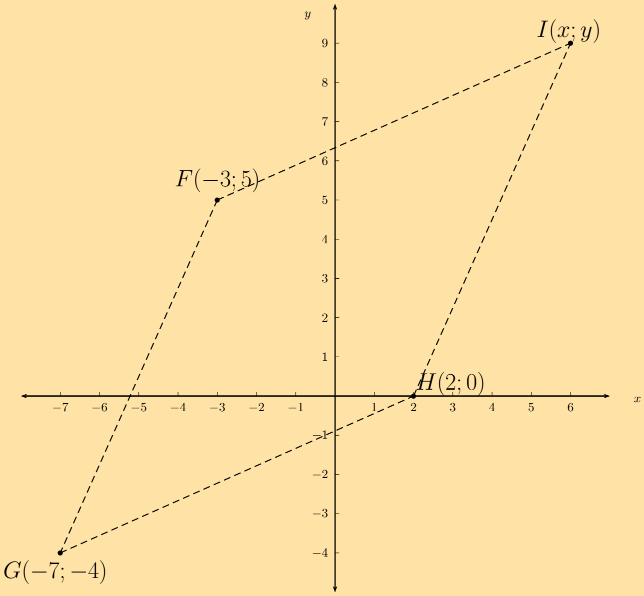
a) Plot the points on the Cartesian plane.

Solution:



b) Determine the coordinates of I if $FGHI$ is a parallelogram.

Solution:



$$\begin{aligned} m_{GH} &= \frac{-4-0}{-7-2} \\ &= \frac{4}{9} \\ \therefore m_{FI} &= \frac{4}{9} \end{aligned}$$

$\therefore$ from $F(-3;5)$: 9 units to the right and 4 units up

$$\therefore I = (6;9)$$

c) Prove that $FGHI$ is a rhombus.

Solution:

$$\begin{aligned} FG &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ &= \sqrt{(-7-(-3))^2+(-4-5)^2} \\ &= \sqrt{(-4)^2+(-9)^2} \\ &= \sqrt{16+81} \\ &= \sqrt{97} \\ HG &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ &= \sqrt{(2+7)^2+(0+4)^2} \\ &= \sqrt{(9)^2+(4)^2} \\ &= \sqrt{81+16} \\ &= \sqrt{97} \\ \therefore FG &= HG \end{aligned}$$

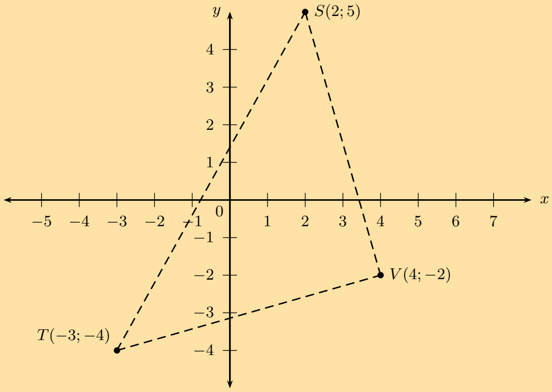
$\therefore FGHI$ is a rhombus (parallelogram with adj. sides equal)

9. Given points $S(2;5)$, $T(-3;-4)$ and $V(4;-2)$.

a) Show that the equation of the line ST is $5y = 9x + 7$.

Solution:

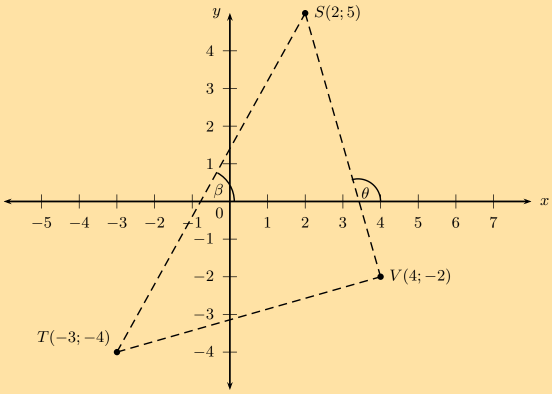
Draw a sketch:



$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{5+4}{2+3} \\
&= \frac{9}{5} \\
y &= mx + c \\
y &= \frac{9}{5}x + c \\
\text{Substitute } S(2;5): \quad 5 &= \frac{9}{5}(2) + c \\
\therefore c &= \frac{7}{5} \\
\therefore y &= \frac{9}{5}x + \frac{7}{5} \\
\therefore 5y &= 9x + 7
\end{aligned}
$$

b) Determine the size of $T\hat{S}V$.

Solution:



Let the angle of inclination of the line ST be β.

Let the angle of inclination of the line SV be θ.

$T\hat{S}V = \theta - \beta$. (exterior $\angle$ of $\triangle$ = sum on opp. interior $\angle$'s)

$$m_{ST} = \frac{9}{5}$$
$$\tan \beta = \frac{9}{5}$$
$$\therefore \beta = \tan^{-1}\left(\frac{9}{5}\right)$$
$$= 60{,}9^\circ$$
$$m_{SV} = \frac{5+2}{2-4}$$
$$= \frac{7}{-2}$$
$$\tan \theta = -\frac{7}{2}$$
$$\therefore \theta = \tan^{-1}\left(-\frac{7}{2}\right)$$
$$= -74{,}1^\circ + 180^\circ$$
$$= 105{,}9^\circ$$
$$\therefore T\hat{S}V = 105{,}9^\circ - 60{,}9^\circ$$
$$= 45^\circ$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 2946 1b. 2947 1c. 2948 1d. 2949 1e. 294B 1f. 294C
1g. 294D 1h. 294F 2. 294G 3. 294H 4. 294J 5. 294K
6. 294M 7. 294N 8. 294P 9. 294Q

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7.2 Equation of a circle

Equation of a circle with centre at the origin

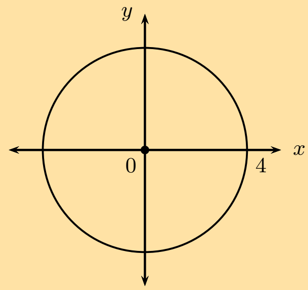
Exercise 7 – 3: Equation of a circle with centre at the origin

1. Complete the following for each circle given below:
   - Determine the radius.
   - Draw a sketch.
   - Calculate the coordinates of two points on the circle.

   a) $x^2 + y^2 = 16$

   Solution:

$$x^2 + y^2 = 16$$
$$r^2 = 16$$
$$\therefore r = 4$$



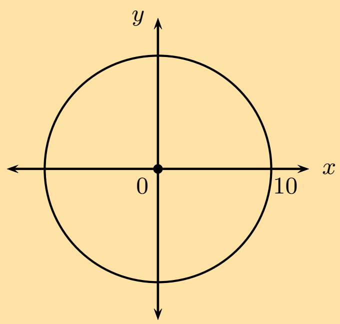
$$
\begin{aligned}
\text{If } x &= 1 \\
(1)^2 + y^2 &= 16 \\
y^2 &= 15 \\
\therefore y &= \pm\sqrt{15}
\end{aligned}
$$

This gives the points $\left(1; \sqrt{15}\right)$ and $\left(1; -\sqrt{15}\right)$.

b) $x^2 + y^2 = 100$

Solution:

$$
\begin{aligned}
x^2 + y^2 &= 100 \\
r^2 &= 100 \\
\therefore r &= 10
\end{aligned}
$$



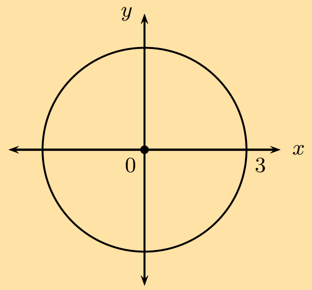
$$
\begin{aligned}
\text{If } x &= 2 \\
(2)^2 + y^2 &= 100 \\
y^2 &= 96 \\
\therefore y &= \pm\sqrt{96}
\end{aligned}
$$

This gives the points $\left(2; \sqrt{96}\right)$ and $\left(2; -\sqrt{96}\right)$.

c) $3x^2 + 3y^2 = 27$

Solution:

$$
\begin{aligned}
3x^2 + 3y^2 &= 27 \\
x^2 + y^2 &= 9 \\
r^2 &= 9 \\
\therefore r &= 3
\end{aligned}
$$



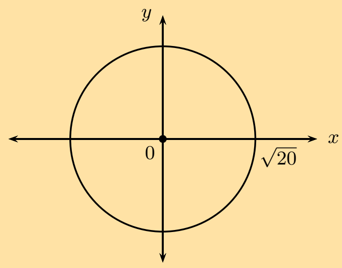
$$\begin{aligned} \text{If } x &= 1 \\ (1)^2 + y^2 &= 9 \\ y^2 &= 8 \\ \therefore y &= \pm\sqrt{8} \end{aligned}$$

This gives the points $(1; \sqrt{8})$ and $(1; -\sqrt{8})$.

d) $y^2 = 20 - x^2$

Solution:

$$\begin{aligned} x^2 + y^2 &= 20 \\ r^2 &= 20 \\ \therefore r &= \sqrt{20} \end{aligned}$$



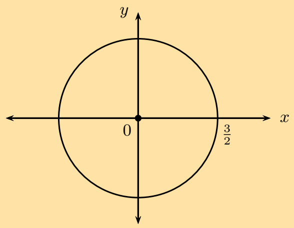
$$\begin{aligned} \text{If } x &= 2 \\ (2)^2 + y^2 &= 20 \\ y^2 &= 16 \\ \therefore y &= \pm 4 \end{aligned}$$

This gives the points $(2; 4)$ and $(2; -4)$.

e) $x^2 + y^2 = 2{,}25$

Solution:

$$\begin{aligned} x^2 + y^2 &= 2{,}25 \\ \text{Convert to a fraction:} \quad 2{,}25 &= \frac{225}{100} = \frac{45}{20} = \frac{9}{4} \\ \therefore x^2 + y^2 &= \frac{9}{4} \\ r^2 &= \frac{9}{4} \\ \therefore r &= \frac{3}{2} \end{aligned}$$

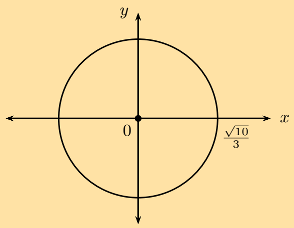


$$\begin{aligned} \text{If } x &= 1 \\ (1)^2 + y^2 &= \frac{9}{4} \\ y^2 &= \frac{5}{4} \\ \therefore y &= \pm\frac{\sqrt{5}}{2} \end{aligned}$$

This gives the points $\left(1; \frac{\sqrt{5}}{2}\right)$ and $\left(1; -\frac{\sqrt{5}}{2}\right)$.

f) $y^2 = -x^2 + \frac{10}{9}$
Solution:

$$\begin{aligned} x^2 + y^2 &= \frac{10}{9} \\ r^2 &= \frac{10}{9} \\ \therefore r &= \frac{\sqrt{10}}{3} \end{aligned}$$



$$\begin{aligned} \text{If } x &= 1 \\ (1)^2 + y^2 &= \frac{10}{9} \\ y^2 &= \frac{1}{9} \\ \therefore y &= \pm\frac{1}{3} \end{aligned}$$

This gives the points $\left(1; \frac{1}{3}\right)$ and $\left(1; -\frac{1}{3}\right)$.

2. Determine the equation of the circle:

a) with centre at the origin and a radius of 5 units.
Solution: $x^2 + y^2 = 25$

b) with centre at $(0; 0)$ and $r = \sqrt{11}$ units.
Solution: $x^2 + y^2 = 11$

c) passing through the point $(3; 5)$ and with centre $(0; 0)$.
Solution:

$$\begin{aligned} x^2 + y^2 &= r^2 \\ (3)^2 + (5)^2 &= r^2 \\ 34 &= r^2 \\ \therefore x^2 + y^2 &= 34 \end{aligned}$$

d) centred at the origin and $r = 2{,}5$ units.

Solution:

$$x^2 + y^2 = r^2$$
$$x^2 + y^2 = (2{,}5)^2$$
$$x^2 + y^2 = \left(\frac{5}{2}\right)^2$$
$$x^2 + y^2 = \frac{25}{4}$$

e) with centre at the origin and a diameter of 30 units.

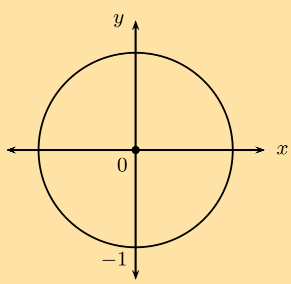
Solution:

$$x^2 + y^2 = r^2$$
$$r = \frac{30}{2} = 15$$
$$x^2 + y^2 = (15)^2$$
$$x^2 + y^2 = 225$$

f) passing through the point $(p; 3q)$ and with centre at the origin.

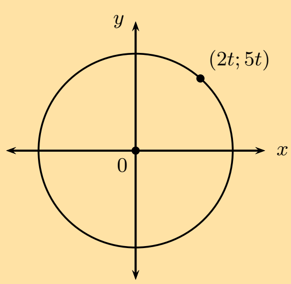
Solution:

$$x^2 + y^2 = r^2$$
$$(p)^2 + (3q)^2 = r^2$$
$$p^2 + 9q^2 = r^2$$
$$\therefore x^2 + y^2 = p^2 + 9q^2$$



g)

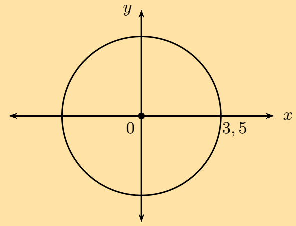
Solution: $x^2 + y^2 = 1$



h)

Solution:

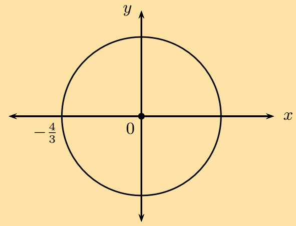
$$x^2 + y^2 = r^2$$
$$(2t)^2 + (5t)^2 = r^2$$
$$4t^2 + 25t^2 = r^2$$
$$29t^2 = r^2$$
$$\therefore x^2 + y^2 = 29t^2$$



i)

Solution:

$$\begin{aligned} x^2 + y^2 &= r^2 \\ x^2 + y^2 &= (3{,}5)^2 \\ x^2 + y^2 &= \left(\frac{7}{2}\right)^2 \\ \therefore x^2 + y^2 &= \frac{49}{4} \end{aligned}$$



j)

Solution:

$$\begin{aligned} x^2 + y^2 &= r^2 \\ x^2 + y^2 &= \left(-\frac{4}{3}\right)^2 \\ \therefore x^2 + y^2 &= \frac{16}{9} \end{aligned}$$

3. Determine whether or not the following equations represent a circle:

   a) $x^2 + y^2 - 8 = 0$

      Solution: Yes: $x^2 + y^2 = 8$

   b) $y^2 - x^2 + 25 = 0$

      Solution: No, cannot be written in the form $x^2 + y^2 = r^2$.

   c) $3x^2 + 6y^2 = 18$

      Solution: No, cannot be written in the form $x^2 + y^2 = r^2$.

   d) $x^2 = \sqrt{6} - y^2$

      Solution: Yes: $x^2 + y^2 = \sqrt{6}$

   e) $y(y + x) = -x(x - y) + 11$

      Solution: Yes: $x^2 + y^2 = 11$

   f) $\sqrt{80} + x^2 - y^2 = 0$

      Solution: No, cannot be written in the form $x^2 + y^2 = r^2$.

   g) $\frac{y^2}{3} + \frac{x^2}{3} = 3$

      Solution: Yes: $x^2 + y^2 = 9$

4. Determine the value(s) of g if $\left(\sqrt{3}; g\right)$ is a point on the circle $x^2 + y^2 = 19$.

   Solution:

$$
\begin{aligned}
x^2 + y^2 &= 19 \\
\left(\sqrt{3}\right)^2 + (g)^2 &= 19 \\
g^2 &= 19 - 3 \\
\therefore g^2 &= 16 \\
\therefore g &= \pm 4
\end{aligned}
$$

This gives the points $\left(\sqrt{3}; 4\right)$ and $\left(\sqrt{3}; -4\right)$.

5. $A(s; t)$ is a point on the circle with centre at the origin and a diameter of 40 cm.

   a) Determine the possible coordinates of A if the value of s is triple the value of t.

   Solution:

$$
\begin{aligned}
d &= 40 \\
r &= \frac{d}{2} \\
&= \frac{40}{2} \\
&= 20
\end{aligned}
$$

$$
\begin{aligned}
s &= 3t \\
x^2 + y^2 &= r^2 \\
(3t)^2 + (t)^2 &= (20)^2 \\
9t^2 + t^2 &= 400 \\
10t^2 &= 400 \\
t^2 &= 40 \\
\therefore t &= \pm\sqrt{40} \\
&= \pm\sqrt{4 \,.\, 10} \\
&= \pm 2\sqrt{10} \\
\therefore s &= 3t \\
&= 3\left(\pm 2\sqrt{10}\right) \\
&= \pm 6\sqrt{10}
\end{aligned}
$$

   Therefore, $A\left(6\sqrt{10}; 2\sqrt{10}\right)$ or $A\left(-6\sqrt{10}; -2\sqrt{10}\right)$

   b) Determine the possible coordinates of A if the value of s is half the value of t.

   Solution:

$$
\begin{aligned}
d &= 40 \\
r &= \frac{d}{2} \\
&= \frac{40}{2} \\
&= 20
\end{aligned}
$$

$$
\begin{aligned}
s &= \frac{t}{2} \\
x^2 + y^2 &= r^2 \\
\left(\frac{t}{2}\right)^2 + (t)^2 &= (20)^2 \\
\frac{t^2}{4} + t^2 &= 400 \\
\frac{t^2}{4} + \frac{4t^2}{4} &= 400 \\
5t^2 &= 1600 \\
t^2 &= 320 \\
\therefore t &= \pm\sqrt{320} \\
&= \pm\sqrt{64 \ . \ 5} \\
&= \pm 8\sqrt{5} \\
\therefore s &= \frac{t}{2} \\
&= \frac{\pm 8\sqrt{5}}{2} \\
&= \pm 4\sqrt{5}
\end{aligned}
$$

Therefore, $A\left(4\sqrt{5}; 8\sqrt{5}\right)$ or $A\left(-4\sqrt{5}; -8\sqrt{5}\right)$

6. $P(-2;3)$ lies on a circle with centre at $(0;0)$.

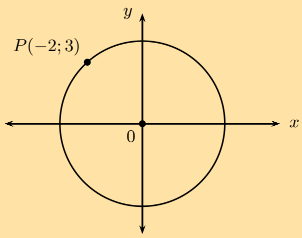
   a) Determine the equation of the circle.

   Solution:

$$
\begin{aligned}
x^2 + y^2 &= r^2 \\
(-2)^2 + (3)^2 &= r^2 \\
4 + 9 &= r^2 \\
13 &= r^2 \\
x^2 + y^2 &= 13
\end{aligned}
$$

   b) Sketch the circle and label point P.

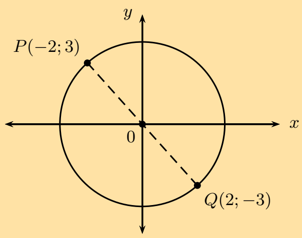
   Solution:



   c) If PQ is a diameter of the circle, determine the coordinates of Q.

   Solution:

   If PQ is a diameter of the circle, then point Q must lie opposite point P on the circumference of the circle. Using symmetry about the origin, the coordinates of point Q are $(2;-3)$.



d) Calculate the length of PQ.

Solution:

$$\begin{aligned} r^2 &= 13 \\ \therefore r &= \sqrt{13} \\ \text{And } d &= 2 \times r \\ &= 2\sqrt{13} \text{ units} \end{aligned}$$

Alternative: Use the distance formula

$$\begin{aligned} PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-2-2)^2 + (3-(-3))^2} \\ &= \sqrt{(-4)^2 + (6)^2} \\ &= \sqrt{16+36} \\ &= \sqrt{52} \\ &= \sqrt{4\ .\ 13} \\ &= 2\sqrt{13} \text{ units} \end{aligned}$$

e) Determine the equation of the line PQ.

Solution:

$$\begin{aligned} m_{PQ} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3-(-3)}{-2-2} \\ &= \frac{6}{-4} \\ &= -\frac{3}{2} \\ y - y_1 &= m(x - x_1) \\ y - y_1 &= -\frac{3}{2}(x - x_1) \\ \text{Substitute } P(-2;3) \quad y - 3 &= -\frac{3}{2}(x-(-2)) \\ y - 3 &= -\frac{3}{2}x - 3 \\ \therefore y &= -\frac{3}{2}x \end{aligned}$$

PQ passes through the origin, therefore $c = 0$.

f) Determine the equation of the line perpendicular to PQ and passing through the point P.

Solution:

For perpendicular line:

$$
\begin{aligned}
m_{PQ} \times m_{\perp} &= -1 \\
-\frac{3}{2} \times m_{\perp} &= -1 \\
m_{\perp} &= \frac{2}{3} \\
y - y_1 &= m(x - x_1) \\
y - y_1 &= \frac{2}{3}(x - x_1) \\
\text{Substitute } P(-2;3) \quad y - 3 &= \frac{2}{3}(x - (-2)) \\
y - 3 &= \frac{2}{3}x + \frac{4}{3} \\
y &= \frac{2}{3}x + \frac{4}{3} + \frac{9}{3} \\
&= \frac{2}{3}x + \frac{13}{3}
\end{aligned}
$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 294S	1b. 294T	1c. 294V	1d. 294W	1e. 294X	1f. 294Y
2a. 294Z	2b. 2952	2c. 2953	2d. 2954	2e. 2955	2f. 2956
2g. 2957	2h. 2958	2i. 2959	2j. 295B	3a. 295C	3b. 295D
3c. 295F	3d. 295G	3e. 295H	3f. 295J	3g. 295K	4. 295M
5. 295N	6. 295P				

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Equation of a circle with centre at $(a;b)$

Exercise 7 – 4: Equation of a circle with centre at $(a;b)$

1. Determine whether or not each of the following equations represents a circle. If not, give a reason.

   a) $x^2 + y^2 + 6y - 10 = 0$
      Solution: Yes
   b) $3x^2 - 35 + 3y^2 = 9y$
      Solution: Yes
   c) $40 = x^2 + 2x + 4y^2$
      Solution: No, coefficients of x^2 term and y^2 term are different.
   d) $x^2 - 4x = \sqrt{21} + 5y + y^2$
      Solution: No, cannot be written in general form $(x-a)^2 + (y-b)^2 = r^2$
   e) $3\sqrt{7} - x^2 - y^2 + 6y - 8x = 0$
      Solution: Yes
   f) $(x-1)^2 + (y+2)^2 + 9 = 0$
      Solution: No, r^2 must be greater than zero.

2. Write down the equation of the circle:

   a) with centre $(0;4)$ and a radius of 3 units.
      Solution: $x^2 + (y-4)^2 = 9$
   b) such that $r = 5$ and the centre is the origin.
      Solution: $x^2 + y^2 = 25$

c) with centre $(-2;3)$ and passing through the point $(4;5)$.
Solution:

$$\begin{aligned}(x-a)^2+(y-b)^2&=r^2\\(x-(-2))^2+(y-3)^2&=r^2\\(x+2)^2+(y-3)^2&=r^2\\\text{Substitute }(4;5):\quad(4+2)^2+(5-3)^2&=r^2\\(6)^2+(2)^2&=r^2\\36+4&=r^2\\40&=r^2\\\therefore(x+2)^2+(y-3)^2&=40\end{aligned}$$

d) with centre $(p;-q)$ and $r=\sqrt{6}$.
Solution: $(x-p)^2+(y+q)^2=6$

e) with $r=\sqrt{10}$ and centre $\left(-\frac{1}{2};\frac{3}{2}\right)$ and
Solution: $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{3}{2}\right)^2=10$

f) with centre $(1;-5)$ and passing through the origin.
Solution:

$$\begin{aligned}(x-a)^2+(y-b)^2&=r^2\\(x-1)^2+(y-(-5))^2&=r^2\\(x-1)^2+(y+5)^2&=r^2\\\text{Substitute }(0;0):\quad(0-1)^2+(0+5)^2&=r^2\\1+25&=r^2\\26&=r^2\\\therefore(x-1)^2+(y+5)^2&=26\end{aligned}$$

3. Determine the centre and the length of the radius for the following circles:

a) $x^2=21-y^2+4y$
Solution:

$$\begin{aligned}x^2&=21-y^2+4y\\x^2+y^2-4y&=21\\x^2+(y^2-4y+4)-4&=21\\x^2+(y-2)^2&=25\end{aligned}$$

Centre: $(0;2)$, $r=5$ units

b) $y^2+x+x^2-\frac{15}{4}=0$
Solution:

$$\begin{aligned}y^2+x+x^2-\frac{15}{4}&=0\\x^2+x+y^2&=\frac{15}{4}\\\left(x^2+x+\frac{1}{4}\right)-\frac{1}{4}+y^2&=\frac{15}{4}\\\left(x+\frac{1}{2}\right)^2+y^2&=\frac{16}{4}\\\left(x+\frac{1}{2}\right)^2+y^2&=4\end{aligned}$$

Centre: $\left(-\frac{1}{2};0\right)$, $r=2$ units

c) $x^2 - 4x + y^2 + 2y - 5 = 0$

Solution:

$$\begin{aligned} x^2 - 4x + y^2 + 2y - 5 &= 0 \\ (x^2 - 4x + 4) - 4 + (y^2 + 2y + 1) - 1 - 5 &= 0 \\ (x-2)^2 + (y+1)^2 - 10 &= 0 \\ (x-2)^2 + (y+1)^2 &= 10 \end{aligned}$$

Centre: $(2; -1)$, $r = \sqrt{10}$ units

d) $x^2 + y^2 - 6y + 2x - 15 = 0$

Solution:

$$\begin{aligned} x^2 + 2x + y^2 - 6y &= 15 \\ (x^2 + 2x + 1) - 1 + (y^2 - 6y + 9) - 9 &= 15 \\ (x+1)^2 + (y-3)^2 &= 25 \end{aligned}$$

Centre: $(-1; 3)$, $r = 5$ units

e) $5 - x^2 - 6x - 8y - y^2 = 0$

Solution:

$$\begin{aligned} 5 - x^2 - 6x - 8y - y^2 &= \\ x^2 + 6x + y^2 + 8y &= 5 \\ (x^2 + 6x + 9) - 9 + (y^2 + 8y + 16) - 16 &= 5 \\ (x+3)^2 + (y+4)^2 &= 30 \end{aligned}$$

Centre: $(-3; -4)$, $r = \sqrt{30}$ units

f) $x^2 - \frac{2}{3}x + y^2 - 4y = \frac{35}{9}$

Solution:

$$\begin{aligned} x^2 - \frac{2}{3}x + y^2 - 4y &= \frac{35}{9} \\ \left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) - \frac{1}{9} + (y^2 - 4y + 4) - 4 &= \frac{35}{9} \\ \left(x - \frac{1}{3}\right)^2 + (y-2)^2 &= \frac{35}{9} + \frac{1}{9} + 4 \\ \left(x - \frac{1}{3}\right)^2 + (y-2)^2 &= 8 \end{aligned}$$

Centre: $\left(\frac{1}{3}; 2\right)$, $r = \sqrt{8}$ units

g) $16x + 2y^2 - 20y + 2x^2 + 42 = 0$

Solution:

$$\begin{aligned} 16x + 2y^2 - 20y + 2x^2 + 42 &= 0 \\ 2x^2 + 16x + 2y^2 - 20y &= -42 \\ x^2 + 8x + y^2 - 10y &= -21 \\ (x^2 + 8x + 16) - 16 + (y^2 - 10y + 25) - 25 &= -21 \\ (x+4)^2 + (y-5)^2 &= 20 \end{aligned}$$

Centre: $(-4; 5)$, $r = \sqrt{20}$ units

h) $6x - 6y - x^2 - y^2 = 6$

Solution:

$$\begin{aligned} 6x - 6y - x^2 - y^2 &= 6 \\ x^2 - 6x + y^2 + 6y &= -6 \\ (x^2 - 6x + 9) - 9 + (y^2 + 6y + 9) - 9 &= -6 \\ (x-3)^2 + (y+3)^2 &= 12 \end{aligned}$$

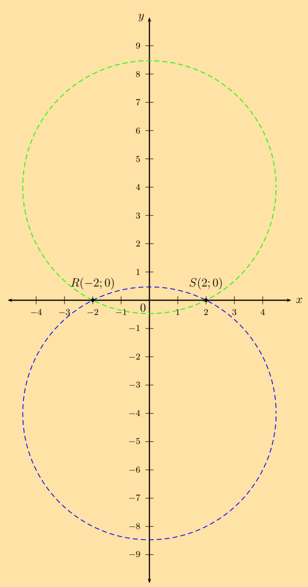
Centre: $(3; -3)$, $r = \sqrt{12}$ units

4. A circle cuts the x-axis at $R(-2; 0)$ and $S(2; 0)$. If $r = \sqrt{20}$ units, determine the possible equation(s) of the circle. Draw a sketch.

Solution:

$$\begin{aligned} (x-a)^2 + (y-b)^2 &= r^2 \\ (x-a)^2 + (y-b)^2 &= 20 \\ \text{Substitute } R(-2;0): \quad (-2-a)^2 + (0-b)^2 &= 20 \\ 4 + 4a + a^2 + b^2 &= 20 \\ 4a + a^2 + b^2 &= 16 \ldots (1) \\ \text{Substitute } S(2;0): \quad (2-a)^2 + (0-b)^2 &= 20 \\ 4 - 4a + a^2 + b^2 &= 20 \\ -4a + a^2 + b^2 &= 16 \ldots (2) \\ (1) - (2): \quad 4a - (-4a) &= 0 \\ 8a &= 0 \\ \therefore a &= 0 \\ \text{And } b^2 &= 16 \\ \therefore b &= \pm 4 \end{aligned}$$

The equation of the circle passing through points R and S is $x^2 + (y+4)^2 = 20$ or $x^2 + (y-4)^2 = 20$.



5. $P(1; 2)$ and $Q(-5; -6)$ are points on a circle such that PQ is a diameter. Determine the equation of the circle.

Solution:

Use the distance formula to determine the length of the diameter.

$$\begin{aligned}
PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(-5-1)^2 + (-6-2)^2} \\
&= \sqrt{(-6)^2 + (-8)^2} \\
&= \sqrt{36 + 64} \\
&= \sqrt{100} \\
&= 10 \\
\text{And } r &= \frac{1}{2} \times \text{ diameter} \\
&= \frac{1}{2} \times 10 \\
&= 5 \\
(x-a)^2 + (y-b)^2 &= (5)^2 \\
(x-a)^2 + (y-b)^2 &= 25
\end{aligned}$$

Given PQ is a diameter of the circle, then the centre is the mid-point of PG:

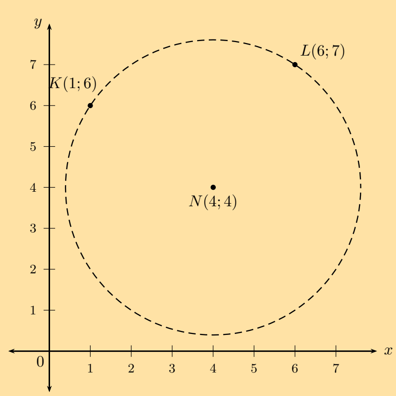
$$\begin{aligned}
M(x;y) &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\
&= \left(\frac{1-5}{2}; \frac{2-6}{2}\right) \\
&= \left(\frac{-4}{2}; \frac{-4}{2}\right) \\
&= (-2;-2)
\end{aligned}$$

Therefore the centre is $(-2;-2)$ and the equation of the circle is $(x+2)^2 + (y+2)^2 = 25$.

6. A circle with centre $N(4;4)$ passes through the points $K(1;6)$ and $L(6;7)$.

   a) Determine the equation of the circle.

   Solution:



$$
\begin{aligned}
(x-a)^2 + (y-b)^2 &= r^2 \\
(x-4)^2 + (y-4)^2 &= r^2 \\
\text{Substitute } K(1;6): \quad (1-4)^2 + (6-4)^2 &= r^2 \\
(-3)^2 + (2)^2 &= r^2 \\
9 + 4 &= r^2 \\
13 &= r^2 \\
\therefore (x-4)^2 + (y-4)^2 &= 13 \\
\text{Or subst. } L(6;7): \quad (6-4)^2 + (7-4)^2 &= r^2 \\
(2)^2 + (3)^2 &= r^2 \\
4 + 9 &= r^2 \\
13 &= r^2 \\
\therefore (x-4)^2 + (y-4)^2 &= 13
\end{aligned}
$$

The equation of the circle is $(x-4)^2 + (y-4)^2 = 13$.

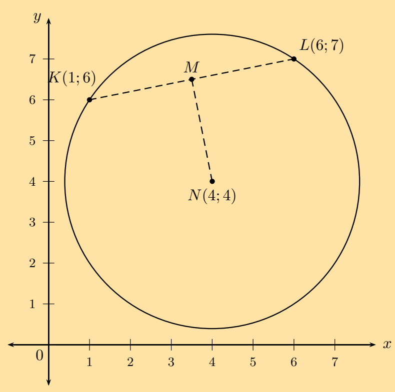
b) Determine the coordinates of M, the mid-point of KL.

Solution:

$$
\begin{aligned}
M(x;y) &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\
&= \left(\frac{1+6}{2}; \frac{6+7}{2}\right) \\
&= \left(\frac{7}{2}; \frac{13}{2}\right)
\end{aligned}
$$

c) Show that $MN \perp KL$.

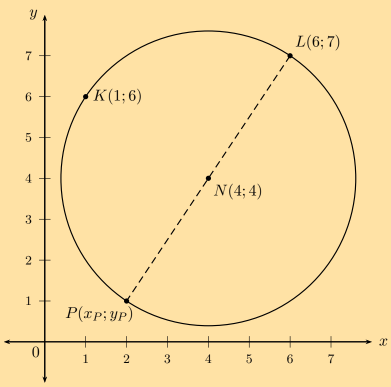
Solution:



$$
\begin{aligned}
m_{KL} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{7-6}{6-1} \\
&= \frac{1}{5} \\
m_{MN} &= \frac{\frac{13}{2} - 4}{\frac{7}{2} - 4} \\
&= \frac{\frac{5}{2}}{-\frac{1}{2}} \\
&= -\frac{5}{2} \times \frac{2}{1} \\
&= -5 \\
\therefore m_{MN} \times m_{KL} &= -5 \times \frac{1}{5} \\
&= -1 \\
\therefore MN &\perp KL
\end{aligned}
$$

d) If P is a point on the circle such that LP is a diameter, determine the coordinates of P.

Solution:



Use the mid-point formula to calculate the coordinates of P:

$$
\begin{aligned}
(x; y) &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2} \right) \\
N(4; 4) &= \left(\frac{x_P + 6}{2}; \frac{y_P + 7}{2} \right)
\end{aligned}
$$

$$
\begin{aligned}
\therefore 4 &= \frac{x_P + 6}{2} \\
8 &= x_P + 6 \\
x_P &= 2 \\
\text{And } 4 &= \frac{y_P + 7}{2} \\
8 &= y_P + 7 \\
y_P &= 1 \\
&\therefore P(2; 1)
\end{aligned}
$$

e) Determine the equation of the line LP.

Solution:

$$\begin{aligned}
m_{LP} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{7-1}{6-2} \\
&= \frac{6}{4} \\
&= \frac{3}{2} \\
y - y_1 &= m(x - x_1) \\
y - y_1 &= \frac{3}{2}(x - x_1) \\
\text{Substitute } L(6;7) y - 7 &= \frac{3}{2}(x - 6) \\
y &= \frac{3}{2}x - 9 + 7 \\
y &= \frac{3}{2}x - 2
\end{aligned}$$

7. A circle passes through the point $A(7;-4)$ and $B(-5;-2)$. If its centre lies on the line $y + 5 = 2x$, determine the equation of the circle.

   Solution:

   Given that the centre of the circle lies on the line $y = 2x - 5$, we can write the coordinates of the circle as $(p; 2p - 5)$ and the equation of the circle becomes:

$$\begin{aligned}
(x-a)^2 + (y-b)^2 &= r^2 \\
(x-p)^2 + (y-(2p-5))^2 &= r^2 \\
(x-p)^2 + (y-2p+5)^2 &= r^2 \\
\text{Substitute } A(7;4): \quad (7-p)^2 + (-4-2p+5)^2 &= r^2 \\
(7-p)^2 + (1-2p)^2 &= r^2 \\
49 - 14p + p^2 + 1 - 4p + 4p^2 &= r^2 \\
5p^2 - 18p + 50 &= r^2 \ldots (1) \\
\text{Substitute } B(-5;-2): \quad (-5-p)^2 + (-2-2p+5)^2 &= r^2 \\
(-5-p)^2 + (3-2p)^2 &= r^2 \\
25 + 10p + p^2 + 9 - 12p + 4p^2 &= r^2 \\
5p^2 - 2p + 34 &= r^2 \ldots (2) \\
(1) - (2): \quad -16p + 16 &= 0 \\
-16p &= -16 \\
\therefore p = x &= 1 \\
\text{And } y &= 2(1) - 5 \\
&= -3 \\
\text{And } r^2 &= 5(1)^2 - 2(1) + 34 \\
&= 5 - 2 + 34 \\
\therefore r^2 &= 37
\end{aligned}$$

   The equation of the circle is $(x-1)^2 + (y+3)^2 = 37$.

8. A circle with centre $(0;0)$ passes through the point $T(3;5)$.

   a) Determine the equation of the circle.

      Solution:

$$x^2 + y^2 = r^2$$
$$(3)^2 + (5)^2 = r^2$$
$$9 + 25 = r^2$$
$$34 = r^2$$
$$\therefore x^2 + y^2 = 34$$

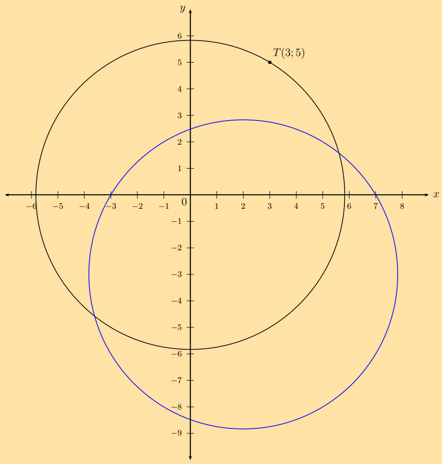
b) If the circle is shifted 2 units to the right and 3 units down, determine the new equation of the circle.

Solution:

$$x^2 + y^2 = 34$$
$$\text{Horizontal shift: } x \text{ is replaced with } x - 2$$
$$\text{Vertical shift: } y \text{ is replaced with } y + 3$$
$$\therefore (x-2)^2 + (y+3)^2 = 34$$

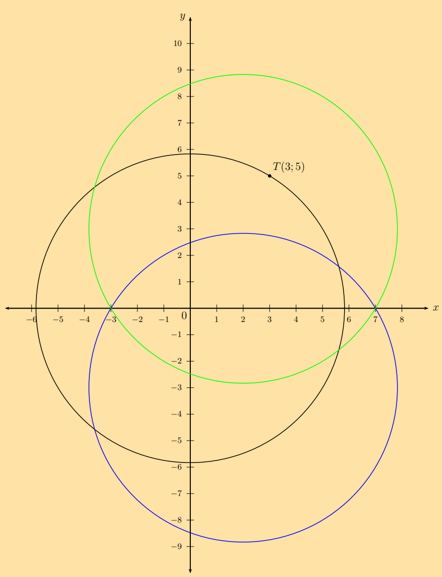
c) Draw a sketch of the original circle and the shifted circle on the same system of axes.

Solution:



d) On the same system of axes as the previous question, draw a sketch of the shifted circle reflected about the x-axis. State the coordinates of the centre of this circle.

Solution:



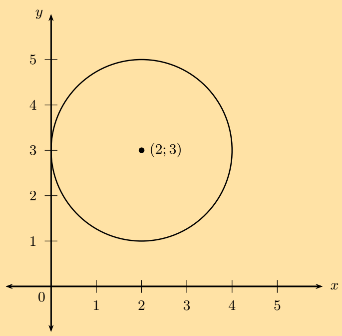
Centre of the shifted circle: $(2; 3)$

9. Determine whether the circle $x^2 - 4x + y^2 - 6y + 9 = 0$ cuts, touches or does not intersect the x-axis and the y-axis.

Solution:

$$\begin{aligned} x^2 - 4x + y^2 - 6y + 9 &= 0 \\ (x-2)^2 - 4 + (y-3)^2 - 9 &= -9 \\ (x-2)^2 + (y-3)^2 &= 4 \\ \therefore (x-2)^2 + (y-3)^2 &= 4 \end{aligned}$$

The radius of the circle is 2 units. The distance from the centre to the y-axis is 2 units, therefore the circle will touch the y-axis. The distance from the centre to the x-axis is 3 units, therefore the circle will not intersect with the x-axis.



Check answers online with the exercise code below or click on 'show me the answer'.

1a. 295Q 1b. 295R 1c. 295S 1d. 295T 1e. 295V 1f. 295W
2a. 295X 2b. 295Y 2c. 295Z 2d. 2962 2e. 2963 2f. 2964
3a. 2965 3b. 2966 3c. 2967 3d. 2968 3e. 2969 3f. 296B
3g. 296C 3h. 296D 4. 296F 5. 296G 6. 296H 7. 296J
8. 296K 9. 296M

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7.3 Equation of a tangent to a circle

Exercise 7 – 5: Equation of a tangent to a circle

1. a) A circle with centre $(8; -7)$ and the point $(5; -5)$ on the circle are given. Determine the gradient of the radius.

Solution:

Given:

- the centre of the circle $(a; b) = (8; -7)$
- a point on the circumference of the circle $(x_1; y_1) = (5; -5)$

Required:

- the gradient of the radius, m

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-5 + 7}{5 - 8} \\ &= -\frac{2}{3} \end{aligned}$$

The gradient of the radius is $m = -\frac{2}{3}$.

b) Determine the gradient of the tangent to the circle at the point $(5; -5)$.

Solution:

The tangent to the circle at the point $(5; -5)$ is perpendicular to the radius of the circle to that same point: $m \times m_{\perp} = -1$

$$\begin{aligned} m_{\perp} &= -\frac{1}{m} \\ &= \frac{-1}{-\frac{2}{3}} \\ &= \frac{3}{2} \end{aligned}$$

The gradient for the tangent is $m_{\perp} = \frac{3}{2}$.

2. Given the equation of the circle: $(x + 4)^2 + (y + 8)^2 = 136$

a) Find the gradient of the radius at the point $(2; 2)$ on the circle.

Solution:

Given:

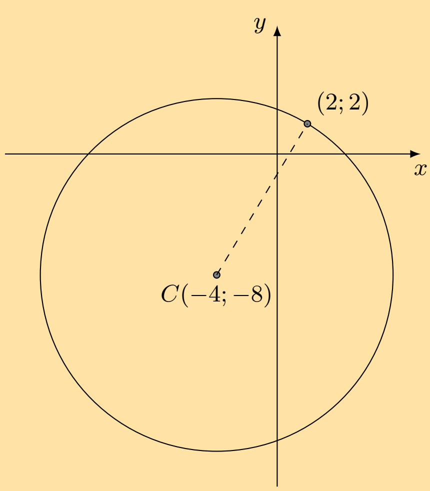
- the equation for the circle $(x + 4)^2 + (y + 8)^2 = 136$
- a point on the circumference of the circle $(x_1; y_1) = (2; 2)$

Required:

- the gradient of the radius, m

The coordinates of the centre of the circle are $(-4; -8)$.

Draw a rough sketch:



$$\begin{aligned} m &= \frac{y_1 - y_0}{x_1 - x_0} \\ &= \frac{2 + 8}{2 + 4} \\ &= \frac{5}{3} \end{aligned}$$

The gradient for this radius is $m = \frac{5}{3}$.

b) Determine the gradient of the tangent to the circle at the point $(2; 2)$.

Solution:

Given:

The tangent to the circle at the point $(2;2)$ is perpendicular to the radius, so $m \times m_{\text{tangent}} = -1$

$$\begin{aligned} m_{\text{tangent}} &= -\frac{1}{m} \\ &= -\frac{1}{\frac{5}{3}} \\ &= -\frac{3}{5} \end{aligned}$$

The gradient for the tangent is $m_{\text{tangent}} = -\frac{3}{5}$.

3. Given a circle with the central coordinates $(a;b) = (-9;6)$. Determine the equation of the tangent to the circle at the point $(-2;5)$.

   Solution:

$$\begin{aligned} m_r &= \frac{y_1 - y_0}{x_1 - x_0} \\ &= \frac{5-6}{-2-(-9)} \\ &= -\frac{1}{7} \end{aligned}$$

   The tangent is perpendicular to the radius, therefore $m \times m_{\perp} = -1$.

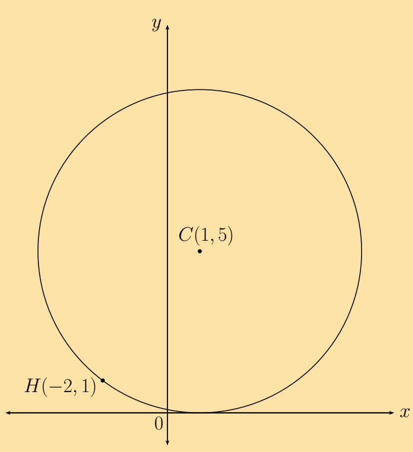
$$\begin{aligned} m &= -\frac{1}{m_r} \\ &= \frac{1}{\frac{1}{7}} \\ &= 7 \end{aligned}$$

   Write down the equation of a straight line and substitute $m = 7$ and $(-2;5)$.

$$\begin{aligned} y_1 &= mx_1 + c \\ 5 &= 7(-2) + c \\ c &= 19 \end{aligned}$$

   The equation of the tangent to the circle is $y = 7x + 19$.

4. Given the diagram below:



   Determine the equation of the tangent to the circle with centre C at point H.

Solution:

Given:

- the centre of the circle $C(a;b) = (1;5)$
- a point on the circumference of the circle $H(-2;1)$

Required:

- the equation for the tangent to the circle in the form $y = mx + c$

Calculate the gradient of the radius:

$$\begin{aligned} m_r &= \frac{y_1 - y_0}{x_1 - x_0} \\ &= \frac{1-5}{-2-1} \\ &= \frac{-4}{-3} \\ &= \frac{4}{3} \end{aligned}$$

$$\begin{aligned} m_r \times m &= -1 \\ m &= -\frac{1}{m_r} \\ &= -\frac{1}{\frac{4}{3}} \\ &= -\frac{3}{4} \end{aligned}$$

Equation of the tangent:

$$\begin{aligned} y &= mx + c \\ 1 &= -\frac{3}{4}(-2) + c \\ 1 &= \frac{3}{2} + c \\ c &= -\frac{1}{2} \end{aligned}$$

The equation for the tangent to the circle at the point H is:

$$y = -\frac{3}{4}x - \frac{1}{2}$$

5. Given the point $P(2;-4)$ on the circle $(x-4)^2 + (y+5)^2 = 5$. Find the equation of the tangent at P.

Solution:

Given:

- the equation for the circle $(x-4)^2 + (y+5)^2 = 5$
- a point on the circumference of the circle $P(2;-4)$

Required:

- the equation of the tangent in the form $y = mx + c$

The coordinates of the centre of the circle are $(a;b) = (4;-5)$.

The gradient of the radius:

$$\begin{aligned} m_r &= \frac{y_1 - y_0}{x_1 - x_0} \\ &= \frac{-4-(-5)}{2-4} \\ &= -\frac{1}{2} \end{aligned}$$

$$
\begin{aligned}
m \times m_{\perp} &= -1 \\
\therefore m_{\perp} &= -\frac{1}{m_r} \\
&= \frac{1}{\frac{1}{2}} \\
&= 2
\end{aligned}
$$

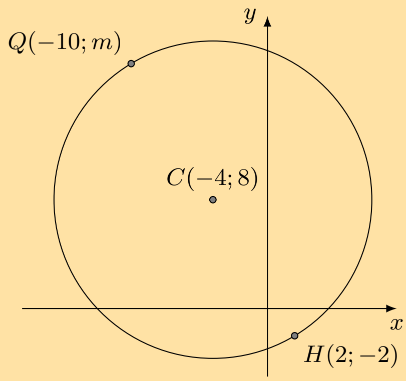
Equation of the tangent:

$$
\begin{aligned}
y &= m_{\perp}x + c \\
-4 &= 2(2) + c \\
c &= -8
\end{aligned}
$$

The equation of the tangent to the circle is

$$y = 2x - 8$$

6. $C(-4;8)$ is the centre of the circle passing through $H(2;-2)$ and $Q(-10;m)$.



a) Determine the equation of the circle.
Solution:
Use the distance formula to determine the length of the radius:

$$
\begin{aligned}
r &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(2+4)^2 + (-2-8)^2} \\
&= \sqrt{(6)^2 + (-10)^2} \\
&= \sqrt{136}
\end{aligned}
$$

Write down the general equation of a circle and substitute r and $H(2;-2)$:

$$
\begin{aligned}
(x-a)^2 + (y-b)^2 &= r^2 \\
(x-(-4))^2 + (y-(8))^2 &= (\sqrt{136})^2 \\
(x+4)^2 + (y-8)^2 &= 136
\end{aligned}
$$

The equation of the circle is $(x+4)^2 + (y-8)^2 = 136$.

b) Determine the value of m.
Solution:
Substitute the $Q(-10;m)$ and solve for the m value.

$$
\begin{aligned}
(x+4)^2 + (y-8)^2 &= 136 \\
(-10+4)^2 + (m-8)^2 &= 136 \\
36 + (m-8)^2 &= 136 \\
m^2 - 16m + 100 &= 136 \\
m^2 - 16m - 36 &= 0 \\
(m+2)(m-18) &= 0
\end{aligned}
$$

The solution shows that $y = -2$ or $y = 18$. From the graph we see that the y-coordinate of Q must be positive, therefore $Q(-10; 18)$.

c) Determine the equation of the tangent to the circle at point Q.

Solution:

Calculate the gradient of the radius:

$$\begin{aligned} m_r &= \frac{y_2 - y_0}{x_2 - x_0} \\ &= \frac{18 - 8}{-10 + 4} \\ &= -\frac{10}{6} \\ &= -\frac{5}{3} \end{aligned}$$

The radius is perpendicular to the tangent, so $m \times m_\perp = -1$.

$$\begin{aligned} m_\perp &= -\frac{1}{m_r} \\ &= \frac{1}{\frac{5}{3}} \\ &= \frac{3}{5} \end{aligned}$$

The equation for the tangent to the circle at the point Q is:

$$\begin{aligned} y &= m_\perp x + c \\ 18 &= \frac{3}{5}(-10) + c \\ 18 &= -6 + c \\ c &= 24 \end{aligned}$$

$$y = \frac{3}{5}x + 24$$

7. The straight line $y = x + 2$ cuts the circle $x^2 + y^2 = 20$ at P and Q.

a) Calculate the coordinates of P and Q.

Solution:

Substitute the straight line $y = x + 2$ into the equation of the circle and solve for x:

$$\begin{aligned} x^2 + y^2 &= 20 \\ x^2 + (x+2)^2 &= 20 \\ x^2 + x^2 + 4x + 4 &= 20 \\ 2x^2 + 4x - 16 &= 0 \\ x^2 + 2x - 8 &= 0 \\ (x-2)(x+4) &= 0 \\ \therefore x &= 2 \text{ or } x = -4 \\ \text{If } x = 2 \quad y &= 2 + 2 = 4 \\ \text{If } x = -4 \quad y &= -4 + 2 = -2 \end{aligned}$$

This gives the points $P(-4; -2)$ and $Q(2; 4)$.

b) Determine the length of PQ.

Solution:

$$\begin{aligned} PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-4-2)^2 + (-2-4)^2} \\ &= \sqrt{(-6)^2 + (-6)^2} \\ &= \sqrt{36+36} \\ &= \sqrt{36\ .\ 2} \\ &= 6\sqrt{2} \end{aligned}$$

c) Determine the coordinates of M, the mid-point of chord PQ.
Solution:

$$\begin{aligned} M(x;y) &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{-4+2}{2}; \frac{-2+4}{2}\right) \\ &= \left(\frac{-2}{2}; \frac{2}{2}\right) \\ &= (-1;1) \end{aligned}$$

d) If O is the centre of the circle, show that $PQ \perp OM$.
Solution:

$$\begin{aligned} m_{PQ} &= \frac{4-(-2)}{2-(-4)} \\ &= \frac{6}{6} \\ &= 1 \end{aligned}$$

$$\begin{aligned} m_{OM} &= \frac{1-0}{-1-0} \\ &= -1 \\ m_{PQ} \times m_{OM} &= -1 \end{aligned}$$

$$\therefore PQ \perp OM$$

e) Determine the equations of the tangents to the circle at P and Q.
Solution:
Tangent at P:
Determine the gradient of the radius OP:

$$\begin{aligned} m_{OP} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2-0}{-4-0} \\ &= \frac{1}{2} \end{aligned}$$

Let the gradient of the tangent at P be m_P. The tangent of a circle is perpendicular to the radius, therefore we can write:

$$\begin{aligned} m_{OP} \times m_P &= -1 \\ \frac{1}{2} \times m_P &= -1 \\ \therefore m_P &= -2 \end{aligned}$$

Substitute $m_P = -2$ and $P(-4;-2)$ into the equation of a straight line.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - y_1 &= -2(x - x_1) \\ \text{Substitute } P(-4;-2): \quad y + 2 &= -2(x + 4) \\ y &= -2x - 8 - 2 \\ &= -2x - 10 \end{aligned}$$

Tangent at Q:
Determine the gradient of the radius OQ:

$$\begin{aligned} m_{OQ} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 0}{2 - 0} \\ &= 2 \end{aligned}$$

Let the gradient of the tangent at Q be m_Q. The tangent of a circle is perpendicular to the radius, therefore we can write:

$$\begin{aligned} m_{OQ} \times m_Q &= -1 \\ 2 \times m_Q &= -1 \\ \therefore m_Q &= -\frac{1}{2} \end{aligned}$$

Substitute $m_Q = -\frac{1}{2}$ and $Q(2;4)$ into the equation of a straight line.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - y_1 &= -\frac{1}{2}(x - x_1) \\ \text{Substitute } Q(2;4): \quad y - 4 &= -\frac{1}{2}(x - 2) \\ y &= -\frac{1}{2}x + 1 + 4 \\ &= -\frac{1}{2}x + 5 \end{aligned}$$

Therefore the equations of the tangents to the circle are $y = -2x - 10$ and $y = -\frac{1}{2}x + 5$.

f) Determine the coordinates of S, the point where the two tangents intersect.

Solution:
Equate the two linear equations and solve for x:

$$\begin{aligned} -2x - 10 &= -\frac{1}{2}x + 5 \\ -4x - 20 &= -x + 10 \\ -3x &= 30 \\ x &= -10 \\ \text{If } x = -10 \quad y &= -2(-10) - 10 \\ &= 10 \end{aligned}$$

This gives the point $S(-10;10)$.

g) Show that $PS = QS$.

Solution:

$$
\begin{aligned}
PS &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(-4 - (-10))^2 + (-2 - 10)^2} \\
&= \sqrt{(6)^2 + (-12)^2} \\
&= \sqrt{36 + 144} \\
&= \sqrt{180}
\end{aligned}
$$

$$
\begin{aligned}
QS &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(2 - (-10))^2 + (4 - 10)^2} \\
&= \sqrt{(12)^2 + (-6)^2} \\
&= \sqrt{144 + 36} \\
&= \sqrt{180}
\end{aligned}
$$

h) Determine the equations of the two tangents to the circle, both parallel to the line $y + 2x = 4$.

Solution:

The tangent at P, $y = -2x - 10$, is parallel to $y = -2x + 4$. To find the equation of the second parallel tangent:

$$
\begin{aligned}
y &= -2x + 4 \\
\therefore m &= -2 \\
\therefore m_{\text{radius}} &= \frac{1}{2} \\
\text{Eqn. of radius: } y &= \frac{1}{2}x \ldots (1) \\
\text{Substitute } (1): \quad x^2 + y^2 &= 20 \\
x^2 + \left(\frac{1}{2}x\right)^2 &= 20 \\
x^2 + \frac{1}{4}x^2 &= 20 \\
\frac{5}{4}x^2 &= 20 \\
x^2 &= 16 \\
x &= \pm 4 \\
\text{If } x = 4, y &= 2 \\
\text{Substitute } (4; 2): \quad y &= -2x + c \\
2 &= -2(4) + c \\
10 &= c \\
y &= -2x + 10
\end{aligned}
$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 296N 2. 296P 3. 296Q 4. 296R 5. 296S 6. 296T 7. 296V

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Exercise 7 – 6: End of chapter exercises

1. Find the equation of the circle:

   a) with centre $(0;5)$ and radius 5

   Solution:

$$(x)^2 + (y-5)^2 = 5^2$$
$$(x)^2 + (y-5)^2 = 25$$

$$\text{Expanded: } x^2 + y^2 - 10y + 25 = 25$$
$$x^2 + y^2 - 10y = 0$$

   b) with centre $(2;0)$ and radius 4

   Solution:

$$(x-2)^2 + y^2 = 16$$

$$\text{Expanded: } x^2 - 4x + 4 + y^2 = 16$$
$$x^2 - 4x + y^2 - 12 = 0$$

   c) with centre $(-5;7)$ and radius 18

   Solution:

$$(x+5)^2 + (y-7)^2 = 18^2$$
$$(x+5)^2 + (y-7)^2 = 324$$

$$\text{Expanded: } x^2 + 10x + 25 + y^2 - 14y + 49 = 324$$
$$x^2 + 10x + y^2 - 14y - 250 = 0$$

   d) with centre $(-2;0)$ and diameter 6

   Solution:

$$(x+2)^2 + y^2 = 9$$

$$\text{Expanded: } x^2 + 4x + 4 + y^2 - 9 = 0$$
$$x^2 + 4x + y^2 - 5 = 0$$

   e) with centre $(-5;-3)$ and radius $\sqrt{3}$

   Solution:

$$(x+5)^2 + (y+3)^2 = 3$$

$$\text{Expanded: } x^2 + 10x + 25 + y^2 + 6y + 9 = 3$$
$$x^2 + 10x + y^2 + 6y + 31 = 0$$

2. a) Find the equation of the circle with centre $(2;1)$ which passes through $(4;1)$.

Solution:

$$\begin{aligned}(x-2)^2+(y-1)^2&=r^2\\(4-2)^2+(1-1)^2&=r^2\\(2)^2+(0)^2&=r^2\\4&=r^2\\\therefore r&=2\\(x-2)^2+(y-1)^2&=4\end{aligned}$$

Expanded: $x^2-4x+y^2-2y+1=0$

b) Where does it cut the line $y=x+1$?

Solution:

$$\begin{aligned}(x-2)^2+(y-1)^2&=4\\(x-2)^2+(x+1-1)^2&=4\\(x-2)^2+(x)^2&=4\\x^2-4x+4+x^2&=4\\2x^2-4x&=0\\x^2-2x&=0\\x(x-2)&=0\\\therefore x&=0 \text{ or } x=2\\\text{If } x&=0, y=1 \quad (0;1)\\\text{If } x&=2, y=3 \quad (2;3)\end{aligned}$$

3. a) Find the equation of the circle with centre $(-3;-2)$ which passes through $(1;-4)$.

Solution:

$$\begin{aligned}(x+3)^2+(y+2)^2&=r^2\\(1+3)^2+(-4+2)^2&=r^2\\(4)^2+(-2)^2&=r^2\\16+4&=r^2\\20&=r^2\\(x+3)^2+(y+2)^2&=20\end{aligned}$$

b) Find the equation of the circle with centre $(3;1)$ which passes through $(2;5)$.

Solution:

$$\begin{aligned}(x-3)^2+(y-1)^2&=r^2\\(2-3)^2+(5-1)^2&=r^2\\(-1)^2+(4)^2&=r^2\\1+16&=r^2\\17&=r^2\\(x-3)^2+(y-1)^2&=17\end{aligned}$$

4. Find the centre and radius of the following circles:

a) $(x+9)^2+(y-6)^2=36$

Solution:

$(-9;6)$, $r=6$ units

b) $\frac{1}{2}(x-2)^2 + \frac{1}{2}(y-9)^2 = 1$

Solution:

$(2;9)$, $r = \sqrt{2}$ unit

c) $(x+5)^2 + (y+7)^2 = 12$

Solution:

$(-5;-7)$, $r = \sqrt{12}$ units

d) $x^2 + (y+4)^2 = 23$

Solution:

$(0;-4)$, $r = \sqrt{23}$ units

e) $3(x-2)^2 + 3(y+3)^2 = 12$

Solution:

$(2;-3)$, $r = 2$ units

5. Find the x and y intercepts of the following graphs:

a) $x^2 + (y-6)^2 = 100$

Solution:

$$\begin{aligned}
x^2 + (y-6)^2 &= 100 \\
\text{Let } x = 0: \quad (y-6)^2 &= 100 \\
y^2 - 12y + 36 &= 100 \\
y^2 - 12y - 64 &= 0 \\
(y-16)(y+4) &= 0 \\
\therefore y &= 16 \text{ or } y = -4 \\
(0;16) &\text{ and } (0;-4)
\end{aligned}$$

$$\begin{aligned}
x^2 + (y-6)^2 &= 100 \\
\text{Let } y = 0: \quad x^2 + (-6)^2 &= 100 \\
x^2 + 36 &= 100 \\
x^2 &= 64 \\
\therefore x &= -8 \text{ or } x = 8 \\
(-8;0) &\text{ and } (8;0)
\end{aligned}$$

b) $(x+4)^2 + y^2 = 16$

Solution:

$$\begin{aligned}
(x+4)^2 + y^2 &= 16 \\
\text{Let } x = 0: \quad (x+4)^2 + y^2 &= 16 \\
4^2 + y^2 &= 16 \\
y^2 &= 0 \\
&\therefore (0;0)
\end{aligned}$$

$$\begin{aligned}
(x+4)^2 + y^2 &= 16 \\
\text{Let } y = 0: \quad (x+4)^2 &= 16 \\
x^2 + 8x + 16 &= 16 \\
x^2 + 8x &= 0 \\
x(x+8) &= 0 \\
\therefore x &= 0 \text{ or } x = -8 \\
(0;0) &\text{ and } (-8;0)
\end{aligned}$$

6. Find the centre and radius of the following circles:

a) $x^2 + 6x + y^2 - 12y = -20$

Solution:

$$\begin{aligned} x^2 + 6x + y^2 - 12y &= -20 \\ (x+3)^2 - 9 + (y-6)^2 - 36 &= -20 \\ (x+3)^2 + (y-6)^2 &= 25 \end{aligned}$$

The centre of the circle is $(-3; 6)$ and $r = 5$ units.

b) $x^2 + 4x + y^2 - 8y = 0$

Solution:

$$\begin{aligned} x^2 + 4x + y^2 - 8y &= 0 \\ (x+2)^2 - 4 + (y-4)^2 - 16 &= 0 \\ (x+2)^2 + (y-4)^2 &= 20 \end{aligned}$$

The centre of the circle is $(-2; 4)$ and $r = \sqrt{20}$ units.

c) $x^2 + y^2 + 8y = 7$

Solution:

$$\begin{aligned} x^2 + y^2 + 8y &= 7 \\ x^2 + (y+4)^2 - 16 &= 7 \\ x^2 + (y+4)^2 &= 23 \end{aligned}$$

The centre of the circle is $(0; -4)$ and $r = \sqrt{23}$ units.

d) $x^2 - 6x + y^2 = 16$

Solution:

$$\begin{aligned} x^2 - 6x + y^2 &= 16 \\ (x-3)^2 - 9 + y^2 &= 16 \\ (x-3)^2 + y^2 &= 25 \end{aligned}$$

The centre of the circle is $(3; 0)$ and $r = 5$ units.

e) $x^2 - 5x + y^2 + 3y = -\frac{3}{4}$

Solution:

$$\begin{aligned} x^2 - 5x + y^2 + 3y &= -\frac{3}{4} \\ \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + \left(y + \frac{3}{2}\right)^2 - \frac{9}{4} &= -\frac{3}{4} \\ \left(x - \frac{5}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 &= \frac{31}{4} \end{aligned}$$

The centre of the circle is $(\frac{5}{2}; -\frac{3}{2})$ and $r = \frac{\sqrt{31}}{2}$ units.

f) $x^2 - 6nx + y^2 + 10ny = 9n^2$

Solution:

$$\begin{aligned} x^2 - 6nx + y^2 + 10ny &= 9n^2 \\ (x-3n)^2 - 9n^2 + (y+5n)^2 - 25n^2 &= 9n^2 \\ (x-3n)^2 + (y+5n)^2 &= 43n^2 \end{aligned}$$

The centre of the circle is $(3n; -5n)$ and $r = \sqrt{43}n$ units.

7. a) Find the gradient of the radius between the point $(4; 5)$ on the circle and its centre $(-8; 4)$.

Solution:

Given:

- the centre of the circle $(a; b) = (-8; 4)$
- a point on the circumference of the circle $(4; 5)$

Required:

- the gradient m of the radius

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{5 - 4}{4 + 8} \\ &= \frac{1}{12} \end{aligned}$$

The gradient for this radius is $m = \frac{1}{12}$.

b) Find the gradient line tangent to the circle at the point $(4; 5)$.

Solution:

The tangent to the circle at the point $(4; 5)$ is perpendicular to the radius of the circle to that same point:

$$\begin{aligned} m_{\perp} &= -\frac{1}{m} \\ &= \frac{-1}{\frac{1}{12}} \\ &= -12 \end{aligned}$$

The gradient for the tangent is $m_{\perp} = -12$.

8. a) Given $(x - 1)^2 + (y - 7)^2 = 10$, determine the value(s) of x if $(x; 4)$ lies on the circle.

Solution:

$$\begin{aligned} (x - 1)^2 + (4 - 7)^2 &= 10 \\ x^2 - 2x + 1 + 9 &= 10 \\ x(x - 2) &= 0 \\ \therefore x &= 0 \text{ or } x = 2 \end{aligned}$$

The points $(0; 4)$ and $(2; 4)$ lie on the circle.

$(0; 4), (2; 4)$

b) Find the gradient of the tangent to the circle at the point $(2; 4)$.

Solution:

$$\begin{aligned} m_r &= \frac{y_1 - y_0}{x_1 - x_0} \\ &= \frac{4 - 7}{2 - 1} \\ &= -3 \end{aligned}$$

$$\begin{aligned} m_{\text{tangent}} &= -\frac{1}{m} \\ &= -\frac{1}{-3} \\ &= \frac{1}{3} \end{aligned}$$

The gradient of the tangent is $m_{\text{tangent}} = \frac{1}{3}$.

$m = \frac{1}{3}$

9. Given a circle with the central coordinates $(a;b) = (-2;-2)$. Determine the equation of the tangent line of the circle at the point $(-1;3)$.

Solution:

$$\begin{aligned} m_r &= \frac{y_1 - y_0}{x_1 - x_0} \\ &= \frac{3 - (-2)}{-1 - (-2)} \\ &= 5 \end{aligned}$$

The radius is perpendicular to the tangent, therefore $m_r \times m_\perp = -1$:

$$m_\perp = -\frac{1}{5}$$

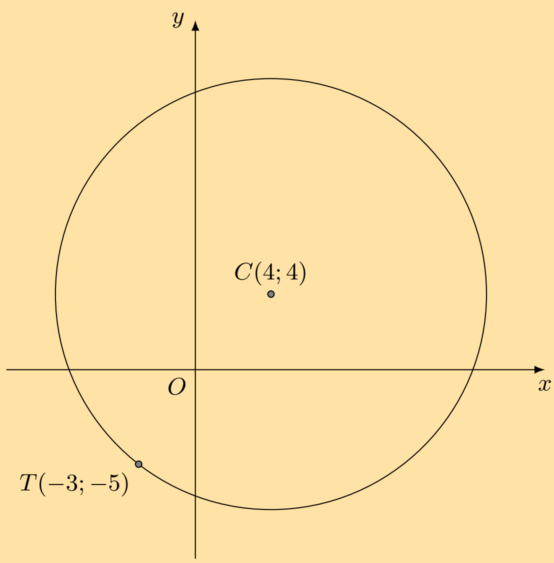
Substitute $m = -\frac{1}{5}$ and $(-1;3)$ to determine c:

$$\begin{aligned} y &= m_\perp x + c \\ 3 &= -\frac{1}{5}(-1) + c \\ c &= \frac{14}{5} \end{aligned}$$

The equation of the tangent to the circle at the point $(-1;3)$ is $y = -\frac{1}{5}x + \frac{14}{5}$.

$y = -\frac{1}{5}x + \frac{14}{5}$

10. Consider the diagram below:



Find the equation of the tangent to the circle at point T.

Solution:

$$\begin{aligned} m_r &= \frac{y_1 - y_0}{x_1 - x_0} \\ &= \frac{4 + 5}{4 + 3} \\ &= \frac{9}{7} \end{aligned}$$

$$m_r \times m = -1$$
$$\therefore m = -\frac{1}{m_r}$$
$$= -\frac{1}{\frac{9}{7}}$$
$$= -\frac{7}{9}$$

Determine the y-intercept (c) of the line by substituting the point $T(-3;-5)$.

$$y = mx + c$$
$$-5 = -\frac{7}{9}(-3) + c$$
$$c = -\frac{22}{3}$$

The equation of the tangent to the circle at T is

$$y = -\frac{7}{9}x - \frac{22}{3}$$

$y = -\frac{7}{9}x - \frac{22}{3}$

11. $M(-2;-5)$ is a point on the circle $x^2 + y^2 + 18y + 61 = 0$. Determine the equation of the tangent at M.

Solution:

Complete the square:

$$x^2 + y^2 + 18y + 61 = 0$$
$$x^2 + (y^2 + 18y) = -61$$
$$x^2 + (y+9)^2 - 81 = -61$$
$$x^2 + (y+9)^2 = 20$$

Therefore the centre of the circle is $(0;-9)$ and $r = \sqrt{20}$ units.

Calculate the gradient of the radius:

$$m_r = \frac{y_1 - y_0}{x_1 - x_0}$$
$$= \frac{-5-(-9)}{-2-0}$$
$$= \frac{4}{-2}$$
$$= -2$$

$$m_{\perp} = -\frac{1}{m_r}$$
$$= -\frac{1}{-2}$$
$$= \frac{1}{2}$$

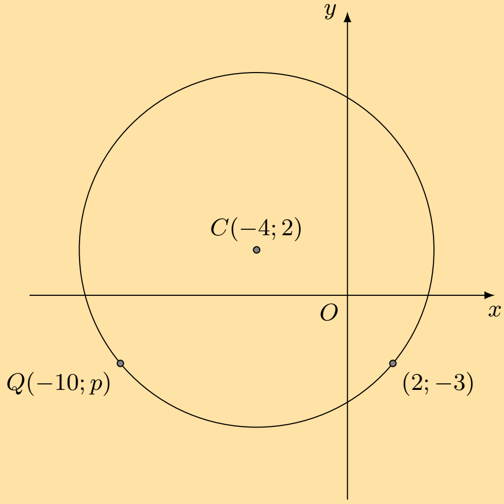
Determine the y-intercept c of the line by substituting the point $M(-2;-5)$.

$$y = m_{\perp}x + c$$
$$-5 = \frac{1}{2}(-2) + c$$
$$c = -4$$

The equation for the tangent to the circle at the point $M(-2;-5)$ is

$$y = \frac{1}{2}x - 4$$

12. $C(-4;2)$ is the centre of the circle passing through $(2;-3)$ and $Q(-10;p)$.



a) Find the equation of the circle given.
Solution:

$$\begin{aligned} r &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\ &= \sqrt{(2+4)^2 + (-3-2)^2} \\ &= \sqrt{(6)^2 + (-5)^2} \\ &= \sqrt{61} \end{aligned}$$

$$\begin{aligned} (x-a)^2 + (y-b)^2 &= r^2 \\ (x-(-4))^2 + (y-(2))^2 &= (\sqrt{61})^2 \\ (x+4)^2 + (y-2)^2 &= 61 \end{aligned}$$

The equation of the circle is $(x+4)^2 + (y-2)^2 = 61$.
$(x+4)^2 + (y-2)^2 = 61$

b) Determine the value of p.
Solution:

$$\begin{aligned} (x+4)^2 + (y-2)^2 &= 61 \\ (-10+4)^2 + (p-2)^2 &= 61 \\ (-10+4)^2 + p^2 - 4p + 4 &= 61 \\ 36 + p^2 - 4p + 4 &= 61 \\ p^2 - 4p - 21 &= 0 \\ (p+3)(p-7) &= 0 \\ \therefore p &= -3 \text{ or } p = 7 \end{aligned}$$

From the graph we see that the correct y-value is -3.
The coordinates for point Q is $Q(-10;-3)$
$p = -3$

c) Determine the equation of the tangent to the circle at point Q.
Solution:

$$\begin{aligned} m_r &= \frac{y_2 - y_0}{x_2 - x_0} \\ &= \frac{-3-2}{-10-(-4)} \\ &= \frac{5}{6} \end{aligned}$$

$m_r \times m_\perp = -1$.

$$\begin{aligned} m_\perp &= -\frac{1}{m_r} \\ &= -\frac{1}{\frac{5}{6}} \\ &= -\frac{6}{5} \end{aligned}$$

Determine the y-intercept c of the line by substituting the point $Q(x_2; y_2) = (-10; -3)$.

$$\begin{aligned} y_2 &= m_\perp x_2 + c \\ -3 &= -\frac{6}{5}(-10) + c \\ c &= -15 \end{aligned}$$

The equation of the tangent to the circle at Q is $y = -\frac{6}{5}x - 15$.

13. Find the equation of the tangent to each circle:

a) $x^2 + y^2 = 17$ at the point $(1; 4)$

Solution:
The centre of the circle is $(0; 0)$ and $r = \sqrt{17}$ units.

$$\begin{aligned} m_r &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 0}{1 - 0} \\ &= 4 \end{aligned}$$

$$\begin{aligned} m_r \times m_\perp &= -1 \\ m_\perp &= -\frac{1}{m_r} \\ &= -\frac{1}{4} \\ &= -\frac{1}{4} \end{aligned}$$

$$\begin{aligned} y_2 &= m_\perp x_2 + c \\ 4 &= -\frac{1}{4}(1) + c \\ c &= \frac{17}{4} \end{aligned}$$

The equation of the tangent to the circle is $y = -\frac{1}{4}x + \frac{17}{4}$.

b) $x^2 + y^2 = 25$ at the point $(3; 4)$

Solution:
The centre of the circle is $(0; 0)$ and $r = 5$ units.

$$\begin{aligned} m_r &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 0}{3 - 0} \\ &= \frac{4}{3} \end{aligned}$$

$$
\begin{aligned}
m_r \times m_\perp &= -1 \\
m_\perp &= -\frac{1}{m_r} \\
&= -\frac{1}{\frac{4}{3}} \\
&= -\frac{3}{4}
\end{aligned}
$$

$$
\begin{aligned}
y_2 &= m_\perp x_2 + c \\
4 &= -\frac{3}{4}(3) + c \\
c &= \frac{25}{4}
\end{aligned}
$$

The equation of the tangent to the circle is $y = -\frac{3}{4}x + \frac{25}{4}$.

c) $(x+1)^2 + (y-2)^2 = 25$ at the point $(3;5)$

Solution:

The centre of the circle is $(-1;2)$ and $r = 5$ units.

$$
\begin{aligned}
m_r &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{5-2}{3-(-1)} \\
&= \frac{3}{4}
\end{aligned}
$$

$$
\begin{aligned}
m_r \times m_\perp &= -1 \\
m_\perp &= -\frac{1}{m_r} \\
&= -\frac{1}{\frac{3}{4}} \\
&= -\frac{4}{3}
\end{aligned}
$$

$$
\begin{aligned}
y_2 &= m_\perp x_2 + c \\
5 &= -\frac{4}{3}(3) + c \\
c &= 9
\end{aligned}
$$

The equation of the tangent to the circle is $y = -\frac{4}{3}x + 9$.

d) $(x-2)^2 + (y-1)^2 = 13$ at the point $(5;3)$

Solution:

The centre of the circle is $(2;1)$ and $r = \sqrt{13}$ units.

$$
\begin{aligned}
m_r &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{3-1}{5-2} \\
&= \frac{2}{3}
\end{aligned}
$$

$$
\begin{aligned}
m_r \times m_\perp &= -1 \\
m_\perp &= -\frac{1}{m_r} \\
&= -\frac{1}{\frac{2}{3}} \\
&= -\frac{3}{2}
\end{aligned}
$$

$$y_2 = m_{\perp}x_2 + c$$
$$3 = -\frac{3}{2}(5) + c$$
$$c = \frac{21}{2}$$

The equation of the tangent to the circle is $y = -\frac{3}{2}x + \frac{21}{2}$.

14. Determine the equations of the tangents to the circle $x^2 + y^2 = 50$, given that both lines have an angle of inclination of $45°$.

Solution:

The centre of the circle is $(0;0)$ and $r = \sqrt{50}$ units.

Gradient of the tangents:

$$\begin{aligned} m &= \tan\theta \\ &= \tan 45° \\ &= 1 \end{aligned}$$

$$\begin{aligned} m \times m_{\perp} &= -1 \\ m &= -1 \end{aligned}$$

The line perpendicular to the tangents and passing through the centre of the circle is $y = -x$. Substitute $y = -x$ into the equation of the circle and solve for x:

$$\begin{aligned} x^2 + (-x)^2 &= 50 \\ 2x^2 &= 50 \\ x^2 &= 25 \\ x &= \pm 5 \end{aligned}$$

This gives the points $P(-5;5)$ and $Q(5;-5)$.

Tangent at $P(-5;5)$:

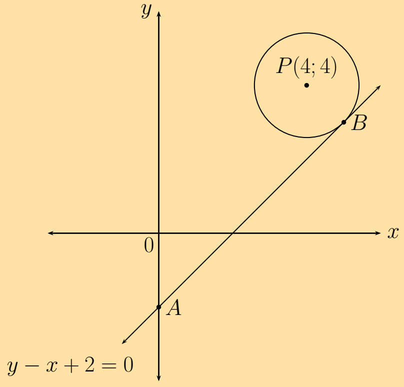
$$\begin{aligned} y - 5 &= (1)(x - (-5)) \\ y &= x + 10 \end{aligned}$$

Tangent at $Q(5;-5)$:

$$\begin{aligned} y - (-5) &= (1)(x - 5) \\ y &= x - 10 \end{aligned}$$

The equations of the tangents to the circle are $y = x - 10$ and $y = x + 10$.

15. The circle with centre $P(4;4)$ has a tangent AB at point B. The equation of AB is $y - x + 2 = 0$ and A lies on the y-axis.



a) Determine the equation of PB.

Solution:

$$\begin{aligned} m_{AB} &= 1 \\ \therefore m_{PB} &= -1 \\ y &= mx + c \\ \text{Substitute } P(4;4): \quad 4 &= -(4) + c \\ \therefore c &= 8 \\ y &= -x + 8 \end{aligned}$$

$y = -x + 8$

b) Determine the coordinates of B.

Solution:

Equate the two equations and solve for x:

$$\begin{aligned} x - 2 &= -x + 8 \\ 2x &= 10 \\ x &= 5 \\ y &= -5 + 8 \\ \therefore y &= 3 \end{aligned}$$

$B(5;3)$

$B(5;3)$

c) Determine the equation of the circle.

Solution:

$$\begin{aligned} PB^2 &= (5-4)^2 + (3-4)^2 \\ &= 1 + 1 \\ &= 2 \\ (x-4)^2 + (y-4)^2 &= 2 \end{aligned}$$

$(x-4)^2 + (y-4)^2 = 2$

d) Describe in words how the circle must be shifted so that P is at the origin.

Solution:

The circle must be shifted 4 units down and 4 units to the left.

e) If the length of PB is tripled and the circle is shifted 2 units to the right and 1 unit up, determine the equation of the new circle.

Solution:

$$\begin{aligned} PB &= \sqrt{2} \\ 3 \times PB &= 3\sqrt{2} \\ \text{Horizontal shift: } (x-4-2)^2 + (y-4)^2 &= \left(3\sqrt{2}\right)^2 \\ (x-6)^2 + (y-4)^2 &= 9(2) \\ \text{Vertical shift: } (x-6)^2 + (y-4-1)^2 &= 18 \\ (x-6)^2 + (y-5)^2 &= 18 \end{aligned}$$

f) The equation of a circle with centre A is $x^2 + y^2 + 5 = 16x + 8y - 30$ and the equation of a circle with centre B is $5x^2 + 5y^2 = 25$. Prove that the two circles touch each other.

Solution:

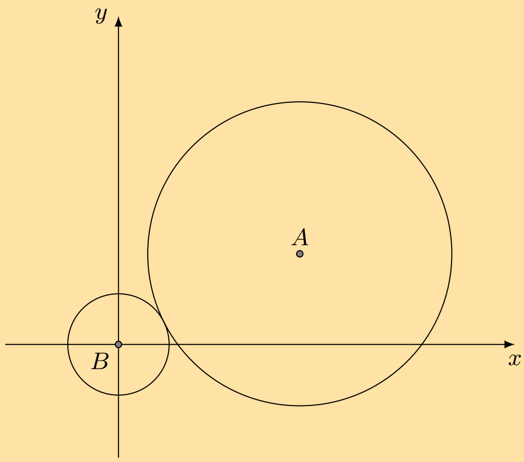
$$\begin{aligned} x^2 + y^2 + 5 &= 16x + 8y - 30 \\ x^2 - 16x + y^2 - 8y &= -35 \\ (x-8)^2 - 64 + (y-4)^2 - 16 &= -35 \\ (x-8)^2 + (y-4)^2 &= 45 \end{aligned}$$

$$\begin{aligned} 5x^2 + 5y^2 &= 25 \\ x^2 + y^2 &= 5 \end{aligned}$$

$$\begin{aligned} AB^2 &= (8-0)^2 + (4-0)^2 \\ &= 64 + 16 \\ &= 80 \\ \therefore AB &= \sqrt{80} \\ AB &= 4\sqrt{5} \end{aligned}$$

$$\begin{aligned} \text{And radius}_A + \text{radius}_B &= \sqrt{45} + \sqrt{5} \\ &= 3\sqrt{5} + \sqrt{5} \\ &= 4\sqrt{5} \\ &= AB \end{aligned}$$

Therefore the two circles touch each other.



Check answers online with the exercise code below or click on 'show me the answer'.

1a. 296W	1b. 296X	1c. 296Y	1d. 296Z	1e. 2972	2. 2973
3. 2974	4a. 2975	4b. 2976	4c. 2977	4d. 2978	4e. 2979
5a. 297B	5b. 297C	6a. 297D	6b. 297F	6c. 297G	6d. 297H
6e. 297J	6f. 297K	7. 297M	8. 297N	9. 297P	10. 297Q
11. 297R	12. 297S	13a. 297T	13b. 297V	13c. 297W	13d. 297X
14. 297Y	15. 297Z				

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CHAPTER 8

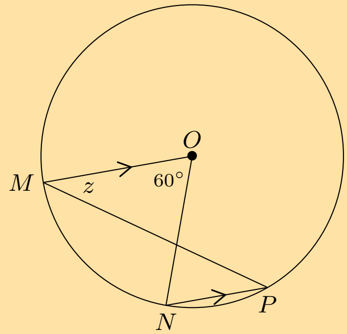
Euclidean geometry

- Sketches are valuable and important tools. Encourage learners to draw accurate diagrams to solve problems.
- It is important to stress to learners that proportion gives no indication of actual length. It only indicates the ratio between lengths.
- To prove triangles are similar, we need to show that two angles (AAA) are equal OR three sides in proportion (SSS).
- Theorems are examinable and are often asked in examinations. It is also important that learners remember the correct construction required for each proof.
- Notation - emphasize to learners the importance of the correct ordering of letters, as this indicates which angles are equal and which sides are in the same proportion.
- If a length has to be calculated from a proportion, it helps to re-write the proportion with the unknown length in the top left position.

8.1 Revision

Exercise 8 – 1: Revision

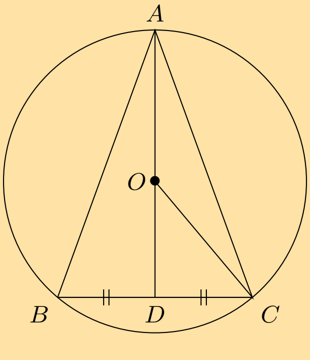
1. $MO \parallel NP$ in a circle with centre O. $M\hat{O}N = 60°$ and $O\hat{M}P = z$. Calculate the value of z, giving reasons.



Solution:

$$\begin{aligned} \hat{P} &= \tfrac{1}{2} M\hat{O}N && (\angle \text{ at centre} = \text{twice } \angle \text{ at circumference}) \\ &= 30° \\ \therefore z &= 30° && (\text{alt. } \angle\text{s, } MO \parallel NP) \end{aligned}$$

2. O is the centre of the circle with $OC = 5$ cm and chord $BC = 8$ cm.



Determine the lengths of:

a) OD

Solution:

$$\text{In } \triangle ODC, \quad OC^2 = OD^2 + DC^2 \quad \text{(Pythagoras)}$$
$$5^2 = OD^2 + 4^2$$
$$\therefore OD = 3 \text{ cm}$$

b) AD

Solution:

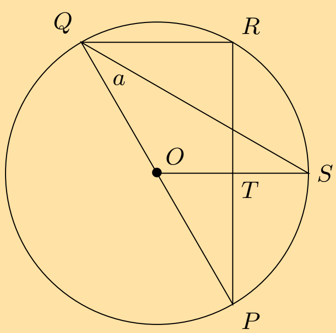
$$AO = 5 \text{ cm} \quad \text{(radius)}$$
$$AD = AO + OD$$
$$= 5 + 3$$
$$\therefore AD = 8 \text{ cm}$$

c) AB

Solution:

$$\text{In } \triangle ABD, \quad AB^2 = BD^2 + AD^2 \quad \text{(Pythagoras)}$$
$$AB^2 = 4^2 + 8^2$$
$$AB = \sqrt{80}$$
$$\therefore AB = 4\sqrt{5}\text{cm}$$

3. PQ is a diameter of the circle with centre O. SQ bisects $P\hat{Q}R$ and $P\hat{Q}S = a$.



a) Write down two other angles that are also equal to a.

Solution:

$$R\hat{Q}S = a \quad \text{(given } SQ \text{ bisects } P\hat{Q}R\text{)}$$
$$OQ = OS \quad \text{(equal radii)}$$
$$\therefore O\hat{Q}S = O\hat{S}Q = a \quad \text{(isosceles } \triangle OQS\text{)}$$

b) Calculate $P\hat{O}S$ in terms of a, giving reasons.

Solution:

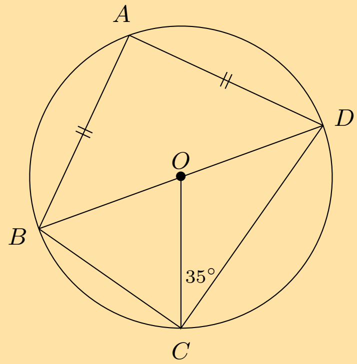
$$P\hat{O}S = 2 \times P\hat{Q}S \quad (\angle\text{s at centre and circumference on same chord})$$
$$= 2a$$

c) Prove that OS is a perpendicular bisector of PR.

Solution:

$$R\hat{Q}S = Q\hat{S}O = a \quad \text{(proven)}$$
$$\therefore QR \parallel OS \quad (\text{alt. } \angle\text{s equal})$$
$$\therefore \hat{R} = R\hat{T}S \quad (\text{alt. } \angle\text{s, } QR \parallel OS)$$
$$= 90^\circ \quad (\hat{R} = \angle \text{ in semi-circle})$$
$$\therefore PT = TR \quad (\perp \text{ from centre bisects chord})$$
$$\therefore OS \text{ perp. bisector of } PR$$

4. BD is a diameter of the circle with centre O. $AB = AD$ and $O\hat{C}D = 35^\circ$.



Calculate the value of the following angles, giving reasons:

a) $O\hat{D}C$
Solution:

$$\begin{aligned} OC &= OD && \text{(equal radii)} \\ \therefore O\hat{D}C &= 35^\circ && \text{(isosceles } \triangle OCD\text{)} \end{aligned}$$

b) $C\hat{O}D$
Solution:

$$\begin{aligned} C\hat{O}D &= 180^\circ - (35^\circ + 35^\circ) && \text{(sum } \angle\text{s } \triangle = 180^\circ\text{)} \\ &= 110^\circ \end{aligned}$$

c) $C\hat{B}D$
Solution:

$$\begin{aligned} C\hat{B}D &= \tfrac{1}{2}C\hat{O}D && (\angle \text{ at centre } = 2\angle \text{ at circum. }) \\ &= 55^\circ \end{aligned}$$

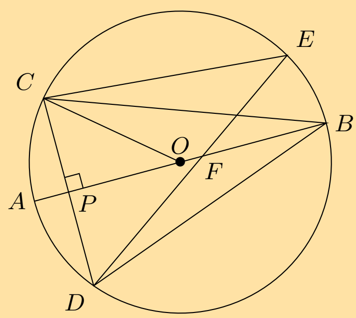
d) $B\hat{A}D$
Solution:

$$B\hat{A}D = 90^\circ \qquad (\angle \text{ in semi-circle})$$

e) $A\hat{D}B$
Solution:

$$\begin{aligned} A\hat{D}B &= A\hat{B}D && \text{(isosceles } \triangle ABD\text{)} \\ \therefore A\hat{D}B &= \tfrac{180^\circ - 90^\circ}{2} && \text{(sum } \angle\text{s in } \triangle = 180^\circ\text{)} \\ &= 45^\circ \end{aligned}$$

5. O is the centre of the circle with diameter AB. $CD \perp AB$ at P and chord DE intersects AB at F.



Prove the following:

a) $C\hat{B}P = D\hat{B}P$

Solution:

In $\triangle CBP$ and $\triangle DBP$:

$CP = DP$	$(OP \perp CD)$
$C\hat{P}B = D\hat{P}B = 90°$	(given)
$BP = BP$	(common)
$\therefore \triangle CBP \equiv \triangle DBP$	(SAS)
$\therefore C\hat{B}P = D\hat{B}P$	$(\triangle CBP \equiv \triangle DBP)$

b) $C\hat{E}D = 2C\hat{B}A$

Solution:

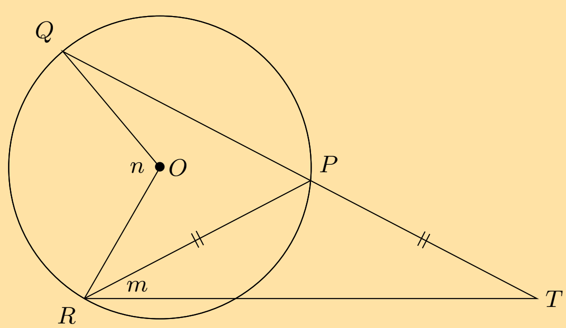
$C\hat{E}D = C\hat{B}D$	($\angle$s on chord CD)
But $C\hat{B}A = D\hat{B}A$	$(\triangle CBP \equiv \triangle DBP)$
$\therefore C\hat{E}D = 2C\hat{B}A$	

c) $A\hat{B}D = \frac{1}{2}C\hat{O}A$

Solution:

$D\hat{B}A = C\hat{B}A$	$(\triangle CBP \equiv \triangle DBP)$
$C\hat{B}A = \frac{1}{2}C\hat{O}A$	($\angle$ at centre $= 2\angle$ at circum.)
$\therefore A\hat{B}D = \frac{1}{2}C\hat{O}A$	

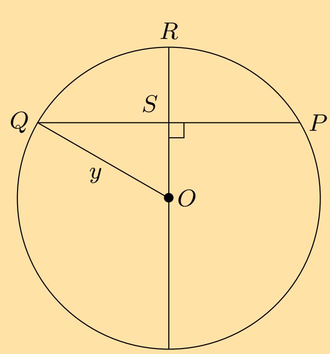
6. QP in the circle with centre O is extended to T so that $PR = PT$. Express m in terms of n.



Solution:

$\hat{T} = m$	$(PT = PR)$
$\therefore Q\hat{P}R = 2m$	(ext. $\angle\triangle =$ sum int. $\angle$s)
$\therefore n = 2(2m)$	($\angle$s at centre and circumference on QR)
$n = 4m$	
$\therefore m = \frac{1}{4}n$	

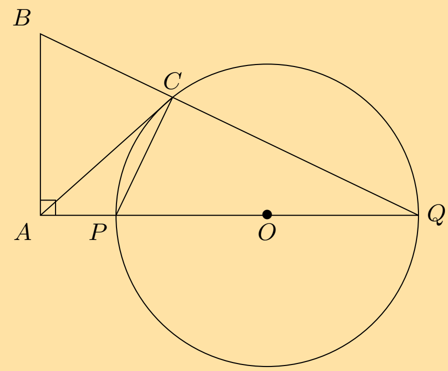
7. In the circle with centre O, $OR \perp QP$, $QP = 30$ mm and $RS = 9$ mm. Determine the length of y.



Solution:

$$\begin{aligned}
&\text{In } \triangle QOS, \\
QP &= 30 && \text{(given)} \\
QS &= \tfrac{1}{2}QP && \text{(}\perp \text{ from centre bisects chord)} \\
\therefore QS &= 15 \\
QO^2 &= OS^2 + QS^2 && \text{(Pythagoras)} \\
y^2 &= (y-9)^2 + 15^2 \\
y^2 &= y^2 - 18y + 81 + 225 \\
\therefore 18y &= 306 \\
\therefore y &= 17 \text{ mm}
\end{aligned}$$

8. PQ is a diameter of the circle with centre O. QP is extended to A and AC is a tangent to the circle. $BA \perp AQ$ and BCQ is a straight line.



Prove the following:

a) $P\hat{C}Q = B\hat{A}P$

Solution:

$$\begin{aligned}
P\hat{C}Q &= 90^\circ && (\angle \text{ in semi-circle}) \\
B\hat{A}Q &= 90^\circ && (\text{given } BA \perp AQ) \\
\therefore P\hat{C}Q &= B\hat{A}Q
\end{aligned}$$

b) $BAPC$ is a cyclic quadrilateral

Solution:

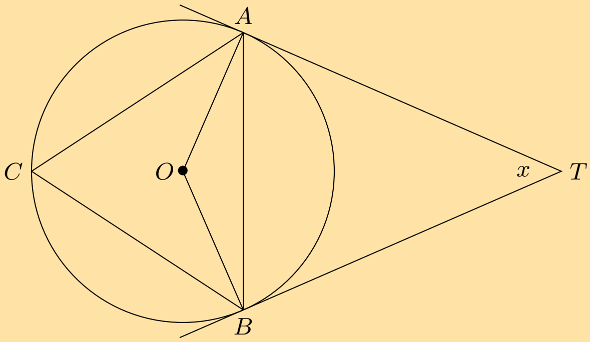
$$\begin{aligned}
P\hat{C}Q &= B\hat{A}Q && \text{(proven)} \\
&\therefore BAPC \text{ is a cyclic quad.} && (\text{ext. angle = int. opp. } \angle)
\end{aligned}$$

c) $AB = AC$

Solution:

$$\begin{aligned}
C\hat{P}Q &= A\hat{B}C && (\text{ext. } \angle \text{ of cyclic quad.}) \\
B\hat{C}P &= C\hat{P}Q + C\hat{Q}P && (\text{ext. } \angle \text{ of } \triangle) \\
A\hat{C}P &= C\hat{Q}P && \text{(tangent-chord)} \\
\therefore B\hat{C}A &= C\hat{P}Q \\
&= A\hat{B}C \\
\therefore AB &= AC && (\angle\text{s opp. equal sides})
\end{aligned}$$

9. TA and TB are tangents to the circle with centre O. C is a point on the circumference and $A\hat{T}B = x$.



Express the following in terms of x, giving reasons:

a) $A\hat{B}T$
Solution:

$$\begin{aligned} A\hat{B}T &= B\hat{A}T && (TA = TB) \\ &= \tfrac{180^\circ - x}{2} && (\text{sum } \angle\text{s of } \triangle TAB) \\ &= 90^\circ - \tfrac{x}{2} \end{aligned}$$

b) $O\hat{B}A$
Solution:

$$\begin{aligned} O\hat{B}T &= 90^\circ && (\text{tangent } \perp \text{ radius}) \\ \therefore O\hat{B}A &= 90^\circ - \left(90^\circ - \tfrac{x}{2}\right) \\ &= \tfrac{x}{2} \end{aligned}$$

c) $\hat{C}$
Solution:

$$\begin{aligned} \hat{C} &= A\hat{B}T && (\text{tangent chord}) \\ &= 90^\circ - \tfrac{x}{2} \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 2982 2. 2983 3. 2984 4. 2985 5. 2986 6. 2987
7. 2988 8. 2989 9. 298B

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8.2 Ratio and proportion

Exercise 8 – 2: Ratio and proportion

1. Solve for p:

a) $\frac{8}{40} = \frac{p}{25}$
Solution:

$$\frac{8}{40} = \frac{p}{25}$$
$$\frac{8 \times 25}{40} = p$$
$$\frac{200}{40} = p$$
$$\therefore 5 = p$$

b) $\frac{6}{9} = \frac{29+p}{54}$

Solution:

$$\frac{6}{9} = \frac{29 + p}{54}$$
$$\frac{6 \times 54}{9} = 29 + p$$
$$36 = 29 + p$$
$$\therefore 7 = p$$

c) $\frac{3}{1+\frac{p}{4}} = \frac{4}{p+1}$

Solution:

$$\frac{3}{1 + \frac{p}{4}} = \frac{4}{p + 1}$$
$$3(p + 1) = 4\left(1 + \frac{p}{4}\right)$$
$$3p + 3 = 4 + p$$
$$2p = 1$$
$$\therefore p = \frac{1}{2}$$

d) $\frac{14}{100-p} = \frac{49}{343}$

Solution:

$$\frac{14}{100 - p} = \frac{49}{343}$$
$$14 \times 343 = 49(100 - p)$$
$$\frac{4802}{49} = 100 - p$$
$$98 = 100 - p$$
$$\therefore p = 2$$

2. A packet of 160 sweets contains red, blue and yellow sweets in the ratio of $3 : 2 : 3$ respectively. Determine how many sweets of each colour there are in the packet.

Solution:

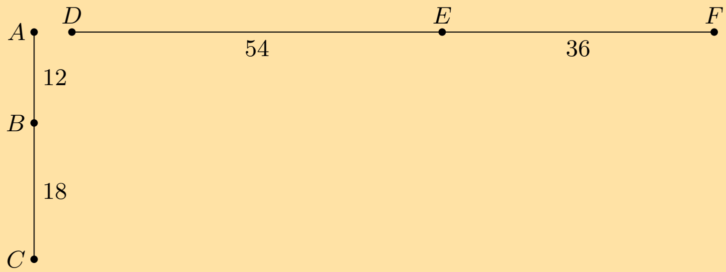
$$3 + 2 + 3 = 8$$
$$\text{Red} = \frac{3}{8} \times 160$$
$$= 60$$
$$\text{Blue} = \frac{2}{8} \times 160$$
$$= 40$$
$$\text{Yellow} = \frac{3}{8} \times 160$$
$$= 60$$

3. A mixture contains 2 parts of substance A for every 5 parts of substance B. If the total weight of the mixture is 50 kg, determine how much of substance B is in the mixture (correct to 2 decimal places).

Solution:

$$\begin{aligned}\text{Ratio substance } A \text{ to substance B } &= 2:5\\ 2+5 &= 7\\ \text{Substance } B &= \frac{5}{7}\times 50\\ &= 35{,}71 \text{ kg}\end{aligned}$$

4. Given the diagram below.



Show that:

a) $\frac{AB}{BC} = \frac{FE}{ED}$

Solution:

$$\begin{aligned}\frac{AB}{BC} &= \frac{12}{18}\\ &= \frac{2}{3}\\ \frac{FE}{ED} &= \frac{36}{54}\\ &= \frac{2}{3}\\ \therefore \frac{AB}{BC} &= \frac{FE}{ED}\end{aligned}$$

b) $\frac{AC}{BC} = \frac{FD}{EF}$

Solution:

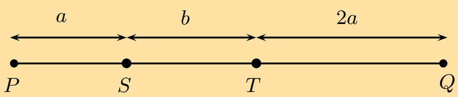
$$\begin{aligned}\frac{AC}{BC} &= \frac{12+18}{18}\\ &= \frac{30}{18}\\ &= \frac{5}{3}\\ \frac{FD}{EF} &= \frac{36+54}{54}\\ &= \frac{90}{54}\\ &= \frac{5}{3}\\ \therefore \frac{AC}{BC} &= \frac{FD}{EF}\end{aligned}$$

c) $AB \,.\, DF = AC \,.\, FE$

Solution:

$$\begin{aligned} AB\ .\ DF &= 12 \times (36+54) \\ &= 12 \times 90 \\ &= 1080 \\ AC\ .\ FE &= (12+18) \times 36 \\ &= 30 \times 36 \\ &= 1080 \\ \therefore AB\ .\ DF &= AC\ .\ FE \end{aligned}$$

5. Consider the line segment shown below.



Express the following in terms of a and b:

a) $PT : ST$

Solution:

$(a+b) : b$

b) $\frac{PS}{TQ}$

Solution:

$\frac{a}{2a} = \frac{1}{2}$

c) $\frac{SQ}{PQ}$

Solution:

$\frac{2a+b}{3a+b}$

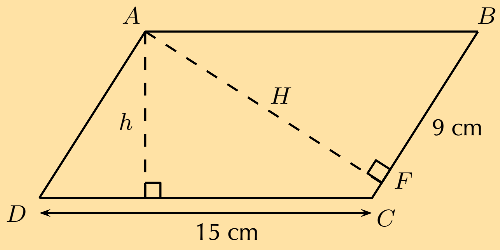
d) $QT : TS$

Solution:

$2a : b$

6. $ABCD$ is a parallelogram with $DC = 15$ cm, $h = 8$ cm and $BF = 9$ cm.

Calculate the ratio $\frac{\text{area } ABF}{\text{area } ABCD}$.



Solution:

The area of a parallelogram $ABCD$ = base × height:

$$\begin{aligned} \text{Area} &= 15 \times 8 \\ &= 120 \text{ cm}^2 \end{aligned}$$

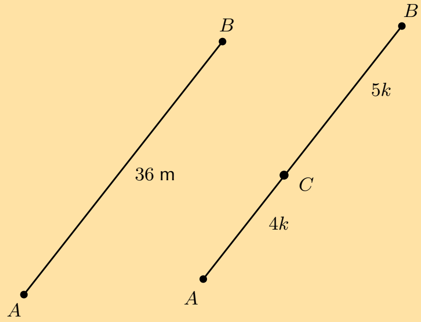
The perimeter of a parallelogram $ABCD = 2DC + 2BC$.

To find the length of BC, we use $AF \perp BC$ and the theorem of Pythagoras.

$$\begin{aligned} \text{In } \triangle ABF\text{:} \quad AF^2 &= AB^2 - BF^2 \\ &= 15^2 - 9^2 \\ &= 144 \\ \therefore AF &= 12 \text{ cm} \\ \therefore \text{area } ABF &= \frac{1}{2} AF \cdot BF \\ &= \frac{1}{2}(12)(9) \\ &= 54 \text{ cm} \end{aligned}$$

$$\begin{aligned} \therefore \frac{\text{area } ABF}{\text{area } ABCD} &= \frac{54}{150} \\ &= \frac{9}{25} \end{aligned}$$

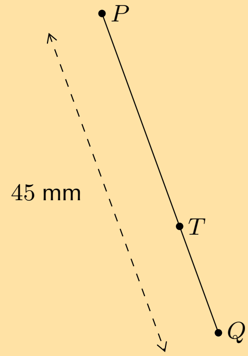
7. $AB = 36$ m and C divides AB in the ratio $4:5$. Determine AC and CB.



Solution:

$$\begin{aligned} \frac{AC}{AB} &= \frac{4k}{(4k+5k)} \\ \therefore AC &= 36 \times \frac{4}{9} \\ &= 16 \text{ m} \\ CB &= 36 - 16 \\ &= 20 \text{ m} \\ \text{Or } \frac{CB}{AC} &= \frac{5k}{(4k+5k)} \\ \therefore CB &= 36 \times \frac{5}{9} \\ &= 20 \text{ m} \end{aligned}$$

8. If $PQ = 45$ mm and the ratio of $TQ : PQ$ is $2:3$, calculate PT and TQ.



Solution:

$$\frac{TQ}{PQ} = \frac{2}{3}$$
$$\therefore TQ = PQ \times \frac{2}{3}$$
$$= 45 \times \frac{2}{3}$$
$$= 30 \text{ mm}$$
$$PT = 45 - 30$$
$$= 15 \text{ mm}$$
$$\text{Or} \quad \frac{PT}{PQ} = \frac{1}{3}$$
$$\therefore PT = 45 \times \frac{1}{3}$$
$$= 15 \text{ mm}$$

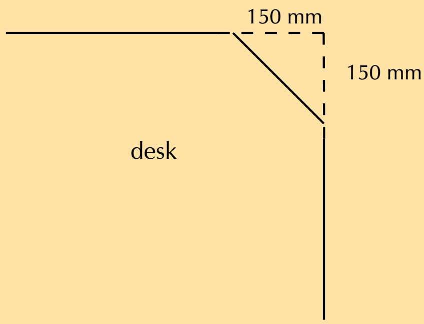
9. Luke's biology notebook is 30 cm long and 20 cm wide. The dimensions of his desk are in the same proportion as the dimensions of his notebook.

   a) If the desk is 90 cm wide, calculate the area of the top of the desk.

   Solution:

$$\text{Ratio} = \frac{\text{table width}}{\text{book width}}$$
$$= \frac{90}{20}$$
$$= 4{,}5$$
$$\therefore \text{Length of table} = 30 \times 4{,}5\text{cm}$$
$$= 135 \text{ cm}$$
$$\text{Area table} = 135 \times 90$$
$$= 12\ 150 \text{ cm}^2$$
$$= 1{,}2 \text{ m}^2$$

   b) Luke covers each corner of his desk with an isosceles triangle of cardboard, as shown in the diagram:



Calculate the new perimeter and area of the visible part of the top of his desk.

Solution:

$$\begin{aligned} x^2 &= 15^2 + 15^2 \\ x &= 21{,}2 \text{ cm} \\ \text{New length} &= 135 - 2(15) \\ &= 105 \text{ cm} \\ \text{New breadth} &= 90 - 2(15) \\ &= 60 \text{ cm} \\ \text{New perimeter} &= 2(105) + 2(60) + 4(21{,}2) \\ &= 414{,}8 \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{Area cut off} &= 2 \times (15^2) \\ &= 450 \text{ m}^2 \\ \text{New area} &= 12\ 150 \text{ cm}^2 - 450 \text{ m}^2 \\ &= 11\ 700 \text{ cm}^2 \end{aligned}$$

c) Use this new area to calculate the dimensions of a square desk with the same desk top area.

Solution:

$$\begin{aligned} s^2 &= 11\ 700 \text{ cm}^2 \\ \therefore s &= 108{,}2 \text{ cm} \\ \text{Square table of length} &\approx 108 \times 108 \text{cm}^2 \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 298C	1b. 298D	1c. 298F	1d. 298G	2. 298H	3. 298J
4a. 298K	4b. 298M	4c. 298N	5a. 298P	5b. 298Q	5c. 298R
5d. 298S	6. 298T	7. 298V	8. 298W	9. 298X	

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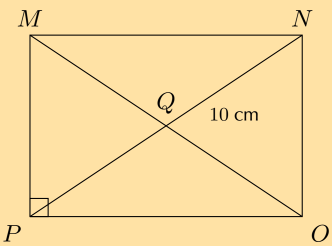
8.3 Polygons

Exercise 8 – 3: Proportionality of polygons

1. $MNOP$ is a rectangle with $MN : NO = 5 : 3$ and $QN = 10$ cm.



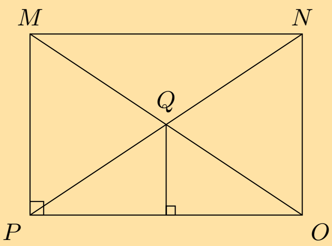
a) Calculate MN (correct to 2 decimal places) .

Solution:

$$\begin{aligned} QN &= 10 \\ \therefore NP &= 2 \times QN && \text{(diagonals bisect each other)} \\ &= 20 \\ \text{In } \triangle NOP, \quad \hat{O} &= 90^\circ && (MNOP \text{ rectangle }) \\ \text{Let } MN &= 5x \\ \text{And } NO &= 3x \\ NP^2 &= NO^2 + OP^2 && \text{(Pythagoras)} \\ (20)^2 &= (3x)^2 + (5x)^2 \\ 400 &= 9x^2 + 25x^2 \\ 400 &= 34x^2 \\ \therefore x &= \sqrt{\tfrac{400}{34}} \\ &= 3{,}43\ldots \\ \therefore MN &= 5x = 17{,}15 \text{ cm} \end{aligned}$$

b) Calculate the area of $\triangle OPQ$ (correct to 2 decimal places).

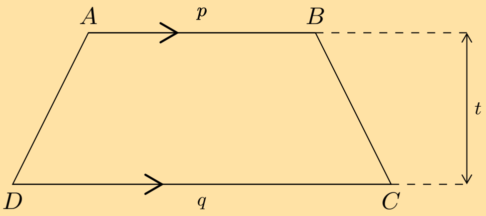
Solution:



$$\begin{aligned} NO &= 3x = 10{,}29 \\ \text{Area } \triangle OPQ &= \frac{1}{2} \text{ base } \times \text{ height} \\ &= \frac{1}{2} MN \times \left(\frac{1}{2} NO\right) \\ &= \frac{1}{2}(17{,}15)\left(\frac{1}{2} \times 10{,}29\right) \\ &= 44{,}12 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{Or} \quad \text{Area } MNOP &= \text{base } \times \text{ height} \\ &= 17{,}15 \times 10{,}29 \\ &= 176{,}43 \text{ cm}^2 \\ \therefore \text{Area } \triangle OPQ &= \frac{1}{4} \times 176{,}43 \\ &= 44{,}12 \text{ cm}^2 \end{aligned}$$

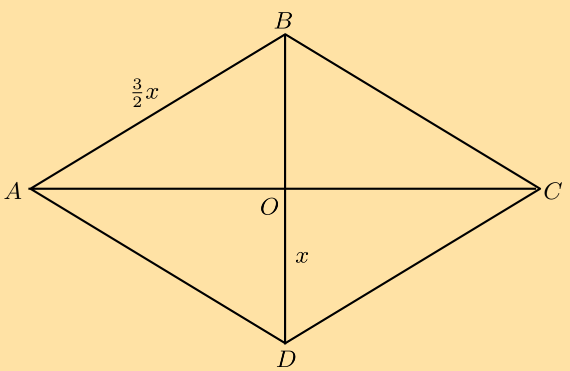
2. Consider the trapezium $ABCD$ shown below. If $t : p : q = 2 : 3 : 5$ and area $ABCD = 288$ cm^2, calculate t, p and q.



Solution:

$$\begin{aligned}
\text{Let } t &= 2x \\
\text{And } p &= 3x \\
\text{And } q &= 5x \\
\text{Area } ABCD &= \frac{1}{2}(p+q) \times t \\
288 &= \frac{1}{2}(3x+5x) \times 2x \\
288 &= 8x^2 \\
36 &= x^2 \\
\therefore 6 &= x \quad \text{(length always positive)} \\
\therefore t &= 2(6) = 12 \text{ cm} \\
p &= 3(6) = 18 \text{ cm} \\
q &= 5(6) = 30 \text{ cm}
\end{aligned}$$

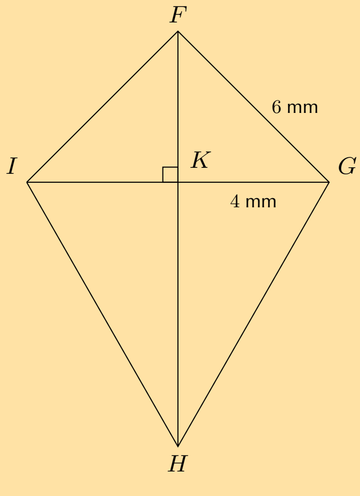
3. $ABCD$ is a rhombus with sides of length $\frac{3}{2}x$ millimetres. The diagonals intersect at O and length $DO = x$ millimetres. Express the area of $ABCD$ in terms of x.



Solution:

$$\begin{aligned} AD &= \frac{3}{2}x \\ DO &= x \\ AO^2 &= \left(\frac{3}{2}x\right)^2 - x^2 \quad (\text{ Pythagoras}) \\ &= \frac{9}{4}x^2 - x^2 \\ &= \frac{5}{4}x^2 \\ \therefore AO &= \frac{x\sqrt{5}}{2} \\ \therefore AC &= x\sqrt{5} \\ \text{Area } &= \frac{1}{2}AC \times BD \\ &= \frac{1}{2} \times x\sqrt{5} \times 2x \\ &= \sqrt{5}x^2 \end{aligned}$$

4. In the diagram below, $FGHI$ is a kite with $FG = 6$ mm, $GK = 4$ mm and $\frac{GH}{FI} = \frac{5}{2}$.



a) Determine FH (correct to the nearest integer).

Solution:

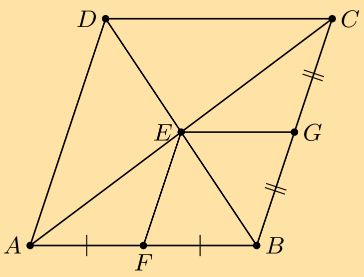
$$
\begin{aligned}
\frac{GH}{FI} &= \frac{5}{2} \\
\text{And } FG &= FI \quad \text{(adj. sides of kite equal)} \\
\frac{GH}{FG} &= \frac{5}{2} \\
\frac{GH}{6} &= \frac{5}{2} \\
\therefore GH &= 15 \text{ mm} \\
\text{In } \triangle FGK, \quad FG^2 &= GK^2 + FK^2 \\
FK^2 &= 6^2 - 4^2 \\
&= 36 - 16 \\
\therefore FK &= \sqrt{20} \\
\text{In } \triangle GKH, \quad GH^2 &= GK^2 + FH^2 \\
15^2 &= 4^2 + KH^2 \\
225 - 16 &= KH^2 \\
\therefore KH &= \sqrt{209} \\
FH &= FK + KH \\
&= \sqrt{20} + \sqrt{209} \\
&= 19 \text{ mm}
\end{aligned}
$$

b) Calculate area $FGHI$.

Solution:

$$
\begin{aligned}
\text{Area } FGHI &= \frac{1}{2} GI \times FH \\
&= \frac{1}{2}(4+4)(19) \\
&= 76 \text{ mm}^2
\end{aligned}
$$

5. $ABCD$ is a rhombus. F is the mid-point of AB and G is the mid-point of CB. Prove that $EFBG$ is also a rhombus.



Solution:

$AF = FB$	(given)
$AE = EC$	(diagonals bisect)
$\therefore FE \parallel BC$	
$\therefore FE \parallel BG$	
$\therefore FE = \frac{1}{2}BG$	(mid-point th.)
$FE = BG$	
$\therefore EFBG$ is a parallelogram	(one pair opp. sides = and $\parallel$)
$\therefore FB \parallel EG$	(opp. sides of parm.)
And $AB = BC$	(adj. sides of rhombus)
$\therefore \frac{1}{2}AB = \frac{1}{2}BC$	
$\therefore FB = BG = GE = EF$	
$\therefore EFBG$ (is a rhombus)	(parm. with 4 equal sides)

Check answers online with the exercise code below or click on 'show me the answer'.
1. 298Y 2. 298Z 3. 2992 4. 2993 5. 2994

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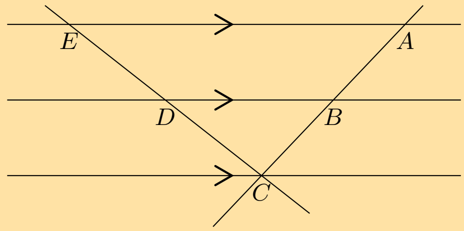
8.4 Triangles

Proportionality of triangles

Exercise 8 – 4: Proportionality of triangles

1. The diagram below shows three parallel lines cut by two transversals EC and AC such that $ED : DC = 4 : 6$.



Determine:

a) $\frac{BC}{AB}$

Solution:

We are given that $ED : DC = 4 : 6$, which we can write as a fraction and simplify:

$$\begin{aligned} \frac{ED}{DC} &= \frac{4}{6} \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \text{And } \frac{ED}{DC} &= \frac{AB}{BC} \\ \frac{AB}{BC} &= \frac{2}{3} \\ \therefore \frac{BC}{AB} &= \frac{3}{2} \end{aligned}$$

b) $AB : AC$

Solution:

$$\begin{aligned} \frac{AB}{AC} &= \frac{ED}{EC} \\ &= \frac{2}{2+3} \\ &= \frac{2}{5} \end{aligned}$$

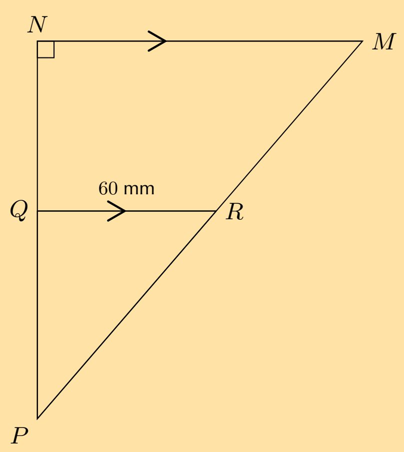
c) The lengths of AC and ED, if it is given that $AB = 12$ mm.

Solution:

$$\begin{aligned}\frac{AB}{AC} &= \frac{ED}{EC} \\ &= \frac{2}{2+3} \\ &= \frac{2}{5} \\ \text{And we know } AB &= 12 \text{ mm} \\ \frac{AB}{AC} &= \frac{2}{5} \times \frac{6}{6} \\ &= \frac{12}{30} \\ \therefore AC &= 30 \text{ mm}\end{aligned}$$

We cannot determine the length of ED since we do not know the lengths of DC, or EC. We only know that $\frac{ED}{EC} = \frac{2}{5}$.

2. In right-angled $\triangle MNP$, QR is drawn parallel to NM, with R the mid-point of MP. $NP = 16$ cm and $RQ = 60$ mm. Determine QP and RP.



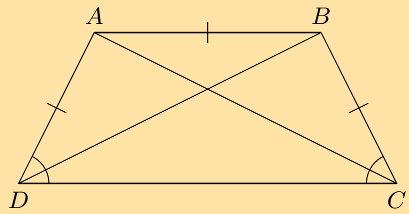
Solution:

$$\begin{aligned}\frac{RP}{MP} &= \frac{QP}{NP} \\ &= \frac{1}{2} \\ \therefore \frac{QP}{16} &= \frac{1}{2} \\ \therefore QP &= 8 \text{ cm}\end{aligned}$$

Use the theorem of Pythagoras to determine RP.

$$\begin{aligned}\text{In } \triangle RQP: \quad PR^2 &= QR^2 + QP^2 \\ &= (8)^2 + (6)^2 \\ &= 64 + 36 \\ &= 100 \\ \therefore PR &= 10 \text{ cm}\end{aligned}$$

3. Given trapezium $ABCD$ with $DA = AB = BC$ and $A\hat{D}C = B\hat{C}D$.



a) Prove that BD bisects $\hat{D}$.

Solution:

$$\begin{aligned} A\hat{D}B &= A\hat{B}D && (AD = AB) \\ A\hat{B}D &= B\hat{D}C && (AB \parallel DC,\ \text{alt. } \angle\text{s}) \\ \therefore A\hat{D}B &= B\hat{D}C \\ \therefore BD &\ \text{bisects } \hat{D} \end{aligned}$$

b) Prove that the two diagonals are equal in length.

Solution:

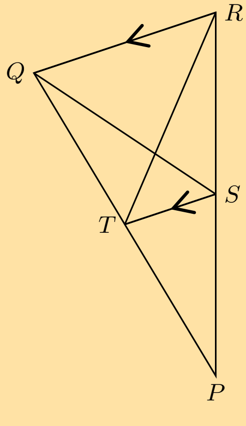
$$\begin{aligned} AD &= BC && \text{(given)} \\ A\hat{D}C &= B\hat{C}D && \text{(given)} \\ DC &= DC \\ \therefore \triangle ADC &\equiv \triangle BCD && \text{(SAS)} \\ \therefore AC &= BD \end{aligned}$$

c) If $DC : AB = 5 : 4$, show that area $ABCD = 2{,}25 \times$ area $\triangle ABC$.

Solution:

$$\begin{aligned} \frac{DC}{AB} &= \frac{5}{4} && \text{(given)} \\ \therefore \frac{\text{Area } \triangle BDC}{\text{Area } \triangle BDA} &= \frac{5}{4} && (\text{same height, } DC \parallel AB) \\ \therefore \frac{\text{Area } ABCD}{\text{Area } \triangle BDA} &= \frac{9}{4} \\ \text{And } \triangle BDA &= \triangle ABC && (\text{same height, same base } AB) \\ \therefore \frac{\text{Area } ABCD}{\text{Area } \triangle ABC} &= 2{,}25 \\ \text{Area } ABCD &= 2{,}25\,(\text{Area } \triangle ABC) \end{aligned}$$

4. In the diagram below, $\triangle PQR$ is given with $QR \parallel TS$.
Show that area $\triangle PQS =$ area $\triangle PRT$.



Solution:

$$\begin{aligned}
\text{Area } \triangle PQS &= \text{Area } \triangle PTS + \text{Area } \triangle SQT \\
\text{Area } \triangle PRT &= \text{Area } \triangle PTS + \text{Area } \triangle SRT \\
\text{Consider } \triangle SQT & \text{ and } \triangle SRT \\
ST & \text{ is common base} \\
QR \parallel TS, & \text{ therefore same height} \\
\therefore \text{Area } \triangle SQT &= \text{Area } \triangle SRT \\
\therefore \text{Area } \triangle PQS &= \text{Area } \triangle PRT
\end{aligned}$$

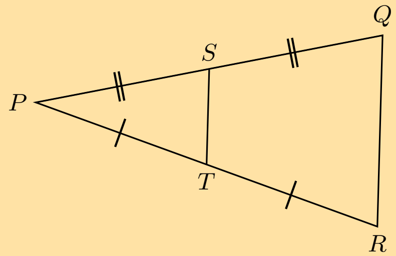
5. In Grade 10 we proved the mid-point theorem using congruent triangles.

   a) Complete the following statement of the mid-point theorem:
   "The line joining . . . . . . of a triangle is . . . . . . to the third side and equal to . . . . . ."

   Solution:

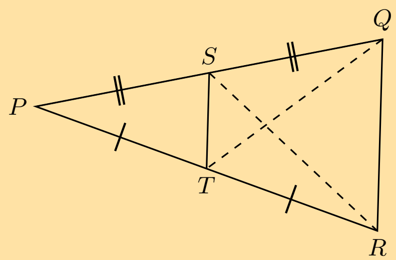
   "The line joining **the mid-points of two sides** of a triangle is **parallel** to the third side and equal to **half the length of the third side.**"

   b) In $\triangle PQR$, T and S are the mid-points of PR and PQ respectively. Prove $TS \parallel RQ$.



   Solution:

   Hint: make a construction by drawing SR and TQ.



$$\begin{aligned}
\text{Area } \triangle SPT &= \text{Area } \triangle SRT && (PT = TR, \text{ and same height}) \\
\text{Area } \triangle SPT &= \text{Area } \triangle SQT && (PS = SQ, \text{ and same height}) \\
\therefore \text{Area } \triangle SRT &= \text{Area } \triangle TQS && (\text{equal area } \triangle SPT) \\
\therefore TS &\parallel RQ && (2\triangle\text{s with the same base } ST \text{ must have the same height})
\end{aligned}$$

   c) Write down the converse of the mid-point theorem.

   Solution:

   Converse: a line through the mid-point of one side of a triangle, parallel to a second side, bisects the third side.

Check answers online with the exercise code below or click on 'show me the answer'.

1\. 2995 2. 2996 3. 2997 4. 2998 5. 2999

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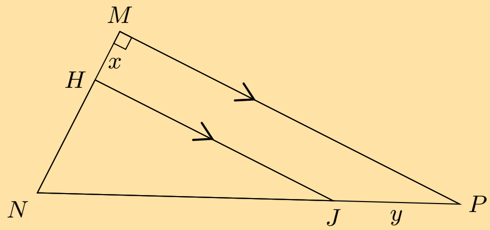
Exercise 8 – 5: Proportion theorem

1. In $\triangle MNP$, $\hat{M} = 90°$ and $HJ \parallel MP$.
   $HN : MH = 3 : 1$, $HM = x$ and $JP = y$.



   a) Calculate $JP : NP$.

   Solution:

$$\begin{aligned}\frac{JP}{NP} &= \frac{HM}{NM} \qquad (HJ \parallel MP)\\ &= \frac{x}{x+3x}\\ &= \frac{x}{4x}\\ &= \frac{1}{4}\end{aligned}$$

   b) Calculate $\frac{\text{area } \triangle HNJ}{\text{area } \triangle MNP}$.

   Solution:

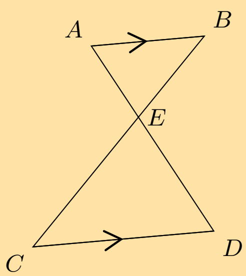
$$\begin{aligned}\frac{HN}{MH} &= \frac{3}{1}\\ \therefore HN &= 3x\\ NJ &= 3y\end{aligned}$$

$$\begin{aligned}\text{In } \triangle HNJ: \quad H &= 90° \qquad (HJ \parallel MP)\\ \therefore HJ^2 &= NJ^2 - NH^2\\ &= (3y)^2 - (3x)^2\\ &= 9y^2 - 9x^2\\ HJ &= \sqrt{9(y^2 - x^2)}\\ &= 3\sqrt{y^2 - x^2}\end{aligned}$$

$$\begin{aligned}\text{In } \triangle MNP: \quad \hat{M} &= 90°\\ MP^2 &= (4y)^2 - (4x)^2\\ MP &= \sqrt{16(y^2 - x^2)}\\ MP &= 4\sqrt{y^2 - x^2}\end{aligned}$$

$$\begin{aligned}\frac{\text{area } \triangle HNJ}{\text{area } \triangle MNP} &= \frac{\frac{1}{2}HJ \times HN}{\frac{1}{2}MP \times MN}\\ &= \frac{3\sqrt{y^2 - x^2} \times 3x}{4\sqrt{y^2 - x^2} \times 4x}\\ &= \frac{9}{16}\end{aligned}$$

2. Use the given diagram to prove the Proportion Theorem.

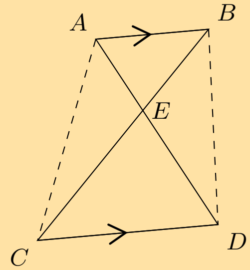


Solution:

Given: $AB \parallel CD$

Required to prove: $AE : ED = BE : EC$

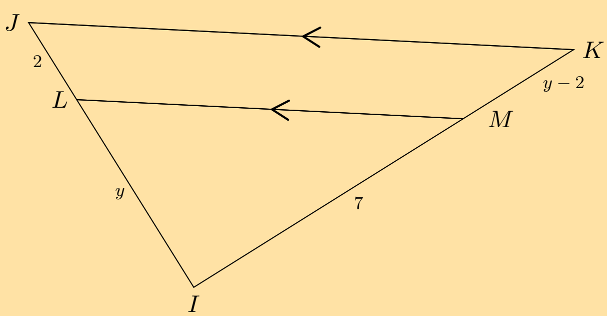
Construction: Draw AC and BD



Proof: use area of $\triangle$'s

$$\begin{aligned}
\frac{\triangle AEB}{\triangle ACB} &= \frac{EB}{CB} && \text{(same height)}\\
\frac{\triangle AEB}{\triangle ADB} &= \frac{AE}{AD} && \text{(same height)}\\
\text{But} \quad \triangle ACB &= \triangle ADB && (AB \parallel CD \quad \therefore \text{ same height})\\
\therefore \frac{EB}{CB} &= \frac{EA}{DA}\\
\therefore \frac{EB}{CE} &= \frac{EA}{DE}
\end{aligned}$$

3. In the diagram below, $JL = 2, LI = y, IM = 7$ and $MK = y - 2$.
   If $LM \parallel JK$, calculate y (correct to two decimal places).



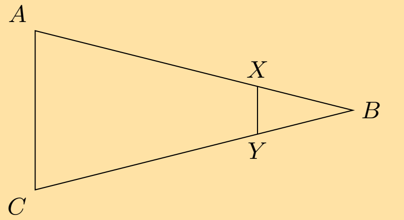
Solution:

$$\begin{aligned}
\frac{LI}{JL} &= \frac{MI}{KM} && (LM \parallel JK)\\
\frac{y}{2} &= \frac{7}{y-2}\\
y(y-2) &= 14\\
y^2 - 2y - 14 &= 0\\
y &= \frac{-(-2) \pm \sqrt{4 - 4(-14)}}{2}\\
&= \frac{2 \pm \sqrt{60}}{2}\\
y &= 4{,}87 \text{ or } -2{,}87\\
\text{But } y &> 0\\
\therefore y &= 4{,}87
\end{aligned}$$

4. Write down the converse of the proportion theorem and illustrate with a diagram.

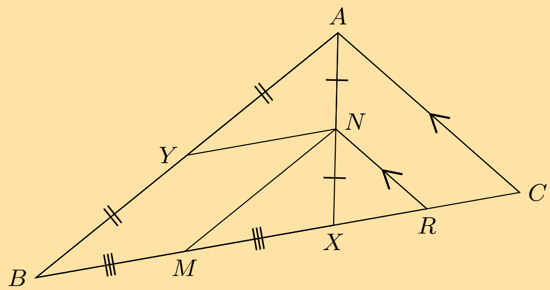
Solution:

Converse of the proportion theorem: a line cutting two sides of a triangle proportionally will be parallel to the third side.



$$\text{If } \frac{AX}{XB} = \frac{CY}{YB}$$
$$\text{then } XY \parallel AC$$

5. In $\triangle ABC$, X is a point on BC. N is the mid-point of AX, Y is the mid-point of AB and M is the mid-point of BX.



a) Prove that $YBMN$ is a parallelogram.

Solution:

Consider $\triangle ABX$:

$YN \parallel BX$	(Y and N mid-points of AB and AX)
$MN \parallel BA$	(M and N mid-points of BX and XA)
$\therefore YBMN$ is a parallelogram	(both opp. sides $\parallel$)

b) Prove that $MR = \frac{1}{2}BC$.

Solution:

Consider $\triangle AXC$:

$RN \parallel CA$	(given)
And $XN = NA$	(N mid-point of AB)
$\therefore XR = RC$	
M is the mid-point BX	(given)
$\therefore MX + XR = \frac{1}{2}BX + \frac{1}{2}XC$	
$\therefore MR = \frac{1}{2}(BX + XC)$	
$MR = \frac{1}{2}(BC)$	

Check answers online with the exercise code below or click on 'show me the answer'.

1. 299B 2. 299C 3. 299D 4. 299F 5. 299G

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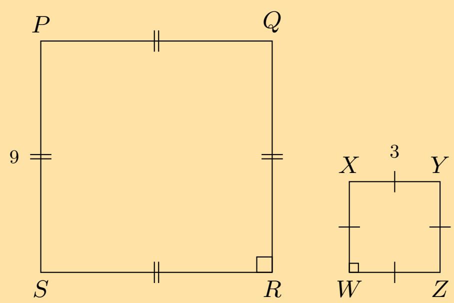
8.5 Similarity

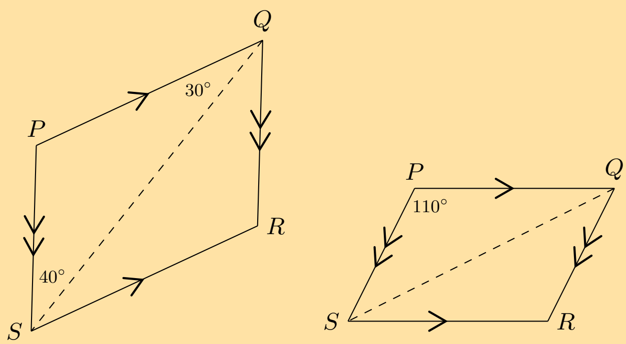
Similar polygons

Exercise 8 – 6: Similar polygons

1. Determine whether or not the following polygons are similar, giving reasons.

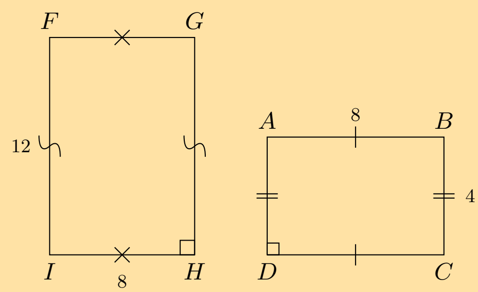
a) 

Solution:
Similar, all angles $= 90°$ and sides are in the same proportion.



b)

Solution:
Not enough information is given. Corresponding angles are equal.



c)

Solution:
Not similar, sides are not in the same proportion.

2. Are the following statements true or false? If false, state reasons or draw an appropriate diagram.

   a) All squares are similar.

   Solution:
   True

b) All rectangles are similar.

Solution:

False. Pairs of corresponding sides not necessarily in the same proportion.

c) All rhombi are similar.

Solution:

False. Corresponding angles not necessarily equal.

d) All congruent polygons are similar.

Solution:

True

e) All similar polygons are congruent.

Solution:

False. Corresponding angles are equal but not necessarily true that corresponding sides are equal.

f) All congruent triangles are similar.

Solution:

True

g) Isosceles triangles are similar.

Solution:

False. Corresponding angles not necessarily equal and pairs of corresponding sides not necessarily in the same proportion.

h) Equilateral triangles are similar.

Solution:

True

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 299H 1b. 299J 1c. 299K 2a. 299M 2b. 299N 2c. 299P
2d. 299Q 2e. 299R 2f. 299S 2g. 299T 2h. 299V

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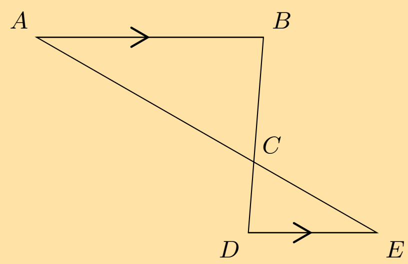
Similarity of triangles

Exercise 8 – 7: Similarity of triangles

1. In the diagram below, $AB \parallel DE$.



a) Prove $\triangle ABC \ ||| \ \triangle EDC$.

Solution:

In $\triangle ABC$ and $\triangle EDC$:

$$\hat{A} = \hat{E} \quad \text{(alt. } \angle\text{s, } AB \parallel DE)$$
$$\hat{B} = \hat{D} \quad \text{(alt. } \angle\text{s, } AB \parallel DE)$$
$$\therefore \triangle ABC \ ||| \ \triangle EDC \quad \text{(AAA)}$$

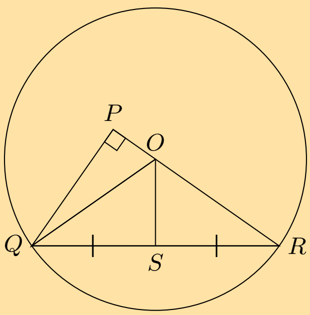
b) If $\frac{AC}{AE} = \frac{5}{7}$ and $AB = 4$ cm, calculate the length of DE (correct to one decimal place).

Solution:

In $\triangle ABC$ and $\triangle EDC$:

$$\frac{AC}{AE} = \frac{5}{7} \qquad \text{(given)}$$
$$\therefore \frac{AC}{CE} = \frac{5}{2}$$
$$\frac{DE}{AB} = \frac{CE}{AC} \qquad (\triangle ABC \ ||| \ \triangle EDC)$$
$$\frac{DE}{4} = \frac{2}{5}$$
$$\therefore DE = \frac{8}{5}$$
$$= 1{,}6 \text{ cm}$$

2. In circle O, $RP \perp PQ$.



a) Prove $\triangle PRQ \ ||| \ \triangle SRO$.

Solution:

In $\triangle PRQ$ and $\triangle SRO$:

$$\hat{P} = 90^\circ \qquad \text{(given)}$$
$$\hat{S} = 90^\circ \qquad (QS = SR)$$
$$\therefore \hat{P} = \hat{S}$$
$$\hat{R} = \hat{R} \qquad \text{(common } \angle)$$
$$\therefore \triangle PRQ \ ||| \ \triangle SRO \qquad \text{(AAA)}$$

b) Prove $\frac{OR}{SR} = \frac{QR}{PR}$.

Solution:

$$\frac{PR}{SR} = \frac{RQ}{RO} = \frac{PQ}{SO} \qquad (\triangle SRO \ ||| \ \triangle PRQ)$$
$$\text{So} \quad \frac{RQ}{RO} = \frac{PR}{SR}$$
$$\frac{QR}{OR} = \frac{PR}{SR}$$
$$QR \, . \, SR = PR \, . \, OR$$
$$\therefore \frac{QR}{PR} = \frac{OR}{SR}$$

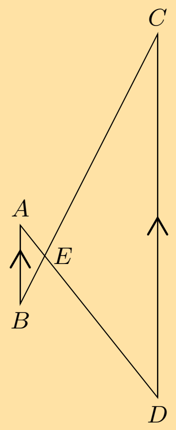
c) If $SR = 18$ mm and $QP = 20$ mm, calculate the radius of circle O (correct to one decimal place).

Solution:

$$\frac{OR}{SR} = \frac{QR}{PR} \quad \text{(proved)}$$

$$\begin{aligned}
\text{In } \triangle PQR: \quad PR^2 &= QR^2 - QP^2 \quad \text{(Pythagoras)} \\
&= (36)^2 - (20)^2 \\
\therefore PR &= \sqrt{896} \\
&= 8\sqrt{14} \\
\therefore \frac{OR}{18} &= \frac{36}{8\sqrt{14}} \\
\therefore OR &= 18 \times \frac{36}{8\sqrt{14}} \\
\therefore \text{radius} &= 21{,}6 \text{ mm}
\end{aligned}$$

3. Given the following figure with the following lengths, find AE, EC and BE.
   $BC = 15$ cm, $AB = 4$ cm, $CD = 18$ cm and $ED = 9$ cm.



Solution:

$$\begin{aligned}
B\hat{A}E &= C\hat{D}E && \text{(alt. } \angle\text{s, } AB \parallel CD) \\
A\hat{B}E &= D\hat{C}E && \text{(alt. } \angle\text{s, } AB \parallel CD) \\
A\hat{E}B &= D\hat{E}C && \text{(vert. opp. } \angle\text{s)} \\
\therefore \triangle AEB &\ ||| \ \triangle DEC && \text{(AAA)}
\end{aligned}$$

$$\therefore \frac{AE}{DE} = \frac{AB}{DC} = \frac{4}{18} = \frac{2}{9} \quad (\triangle AEB \ ||| \ \triangle DEC)$$

$$\begin{aligned}
AE &= \tfrac{2}{9} DE \\
&= \tfrac{2}{9}(9) \\
&= 2 \text{ cm}
\end{aligned}$$

$$\frac{EC}{BC} = \frac{ED}{AD} = \frac{9}{11} \quad (AB \parallel CD)$$

$$\begin{aligned}
EC &= \frac{ED}{AD}(BC) \\
&= \tfrac{9}{11}(15) = 12{,}3 \text{ cm}
\end{aligned}$$

$$\frac{BE}{BC} = \frac{AE}{AD} = \frac{2}{11} \quad (AB \parallel CD)$$

$$\begin{aligned}
BE &= \frac{AE}{AD}(BC) \\
&= \tfrac{2}{11}(15) = 2{,}7 \text{ cm}
\end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 299W 2. 299X 3. 299Y

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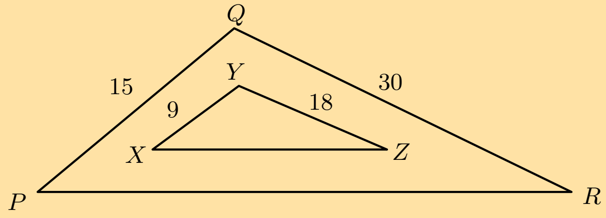
Exercise 8 – 8: Similarity of triangles

1. Consider the diagram given below. $PR = 20$ units and $XZ = 12$ units. Is $\triangle XYZ \ ||| \ \triangle PQR$? Give reasons.

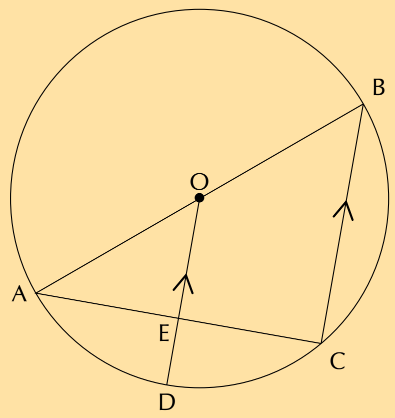


Solution:

$$\frac{XY}{PQ} = \frac{9}{15} = \frac{3}{5}$$
$$\frac{YZ}{QR} = \frac{18}{30} = \frac{3}{5}$$
$$\frac{ZX}{RP} = \frac{12}{20} = \frac{3}{5}$$

Yes, $\triangle XYZ \ ||| \ \triangle PQR$.

2. AB is a diameter of the circle $ABCD$. OD is drawn parallel to BC and meets AC in E. If the radius is 10 cm and $AC = 16$ cm, calculate the length of ED.
[NCS, Paper 3, November 2011]



Solution:

$\hat{C} = 90°$	($\angle$s in semi-circle)
$O\hat{E}A = 90°$	(corresp. $\angle$s; $OD \parallel BC$)
$AE = 8$ cm	(line from centre $\perp$ chord)
$OE = 6$ cm	(Pythagoras)
$ED = 10 - 6 = 4$ cm	

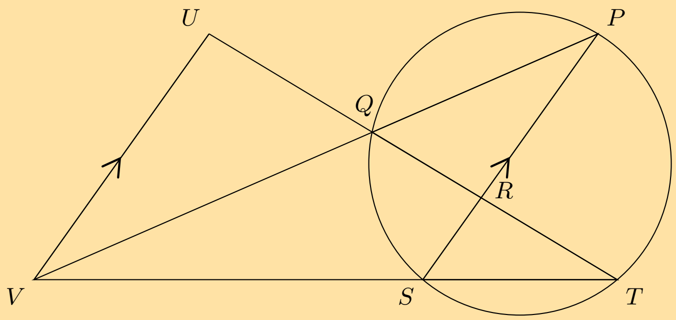
OR i

$$\begin{aligned}
\hat{C} &= 90^\circ && (\angle\text{s in semi-circle})\\
O\hat{E}A &= 90^\circ && (\text{corresp. } \angle\text{s}; OD \parallel BC)\\
OE &\parallel BC && (\text{given})\\
OA &= OB && (\text{radii})\\
AE &= EC = 8 \text{ cm} && (\text{midpoint th.})\\
OE &= 6 \text{ cm} && (\text{Pythagoras})\\
ED &= 10 - 6 = 4 \text{ cm}
\end{aligned}$$

OR

$$\begin{aligned}
\hat{C} &= 90^\circ && (\angle\text{s in semi-circle})\\
BC^2 &= (20)^2 - (16)^2\\
BC^2 &= 144\\
BC &= 12\\
OE &= \tfrac{1}{2}BC && (\text{midpoint th.})\\
OE &= 6 \text{ cm}\\
OD &= 10 \text{ cm}\\
ED &= 10 - 6 = 4 \text{ cm}
\end{aligned}$$

3. P, Q, S and T are on the circumference of the circle.
TS is produced to V so that $SV = 2TS$.
TRQ is produced to U so that $VU \parallel SRP$.



Prove, with reasons, that:

a) $\dfrac{TR}{RU} = \dfrac{1}{3}$

Solution:

$$\begin{aligned}
\frac{TR}{TU} &= \frac{TS}{TV} && (SR \parallel VU)\\
&= \frac{1}{3} && (SV = 2TS)
\end{aligned}$$

b) $\triangle TQV \; ||| \; \triangle PSV$

Solution:

In $\triangle TQV$ and $\triangle PSV$:

$$\begin{aligned}
\hat{P} &= \hat{T} && (\angle\text{s subtended chord } QS)\\
P\hat{V}T &= P\hat{V}T && (\text{common } \angle)\\
\therefore \triangle TQV \; &||| \; \triangle PSV && (\text{AAA})
\end{aligned}$$

c) $QV \,.\, PV = 6TS^2$

Solution:

$$\begin{aligned}
\frac{QV}{TV} &= \frac{SV}{PV} && (\triangle TQV \; ||| \; \triangle PSV)\\
\therefore QV \,.\, PV &= TV.SV\\
\text{But } SV &= 2TS && (\text{given})\\
\therefore TV &= 3TS\\
\therefore QV.PV &= 3TS.2TS\\
QV.PV &= 6TS^2
\end{aligned}$$

d) $\triangle UQV \ ||| \ \triangle RQP \ ||| \ \triangle RST$

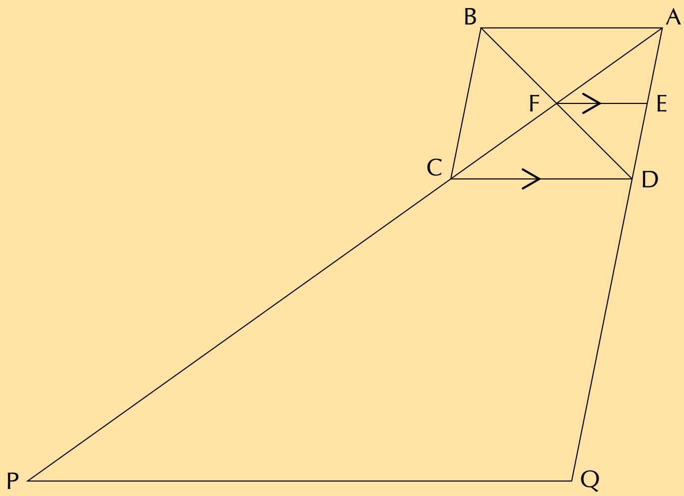
Solution:

In $\triangle UQV$ and $\triangle RQP$ and $\triangle RST$:

$$\begin{aligned} \hat{U} &= Q\hat{R}P && \text{(alt. } \angle\text{s, } VU \parallel RP) \\ &= S\hat{R}T && \text{(vert. opp. } \angle\text{s)} \\ U\hat{V}Q &= \hat{P} && \text{(alt. } \angle\text{s, } VU \parallel RP) \\ &= \hat{T} && (\angle\text{s on chord } QS) \\ \triangle UQV \ ||| \ \triangle RQP \ ||| \ \triangle RST && & \text{(AAA)} \end{aligned}$$

4. $ABCD$ is a parallelogram with diagonals intersecting at F. FE is drawn parallel to CD. AC is produced to P such that $PC = 2AC$ and AD is produced to Q such that $DQ = 2AD$.

[NCS, Paper 3, November 2011]



a) Show that E is the midpoint of AD.

Solution:

$$\begin{aligned} AF &= FC && \text{(diag. parm. bisect)} \\ FE &\parallel CD \\ AE &= ED && \text{(prop th. } FE \parallel CD) \end{aligned}$$

b) Prove $PQ \parallel FE$.

Solution:

$$\begin{aligned} \frac{AC}{CP} &= \frac{1}{2} && \text{(given)} \\ \frac{AD}{DQ} &= \frac{1}{2} && \text{(given)} \\ \frac{AC}{CP} &= \frac{AD}{DQ} \\ CD &\parallel PQ && \text{(converse prop. th.)} \\ CD &\parallel FE && \text{(given)} \\ \therefore PQ &\parallel FE \end{aligned}$$

OR

$$\begin{aligned} \frac{AC}{AP} &= \frac{1}{3} \\ \frac{AD}{AQ} &= \frac{1}{3} \\ \frac{AC}{AP} &= \frac{AD}{AQ} \\ CD &\parallel PQ && \text{(converse prop. th.)} \\ CD &\parallel FE && \text{(given)} \\ \therefore PQ &\parallel FE \end{aligned}$$

OR

$$\frac{AF}{AP} = \frac{1}{6}$$
$$\frac{AE}{AQ} = \frac{1}{6}$$
$$\frac{AF}{AP} = \frac{AE}{AQ}$$
$$\therefore PQ \parallel FE \quad \text{(converse prop. theorem)}$$

c) If PQ is 60 cm, calculate the length of FE.

Solution:

In $\triangle AEF$ and $\triangle APQ$:

i. $\hat{A}$ (is common)
ii. $A\hat{E}F = A\hat{Q}P$ (corresp. $\angle$s, $FE \parallel PQ$)
iii. $A\hat{F}E = A\hat{P}Q$ (corresp. $\angle$s, $FE \parallel PQ$)

$\therefore \triangle BHD \; ||| \; \triangle FED \quad (\angle\angle\angle)$

$$\frac{FE}{PQ} = \frac{AF}{AP} \quad (||| \; \triangle\text{s})$$
$$\frac{FE}{60} = \frac{1}{6}$$
$$FE = 10 \text{ cm}$$

OR

In $\triangle ADC$ and $\triangle APQ$:

i. $\hat{A}$ (is common)
ii. $A\hat{D}C = A\hat{Q}P$ (corresp. $\angle$s, $CD \parallel PQ$)
iii. $A\hat{C}D = A\hat{P}Q$ (corresp. $\angle$s, $CD \parallel PQ$)

$\therefore \triangle ADC \; ||| \; \triangle APQ \quad (\angle\angle\angle)$

$$\frac{AC}{AP} = \frac{AD}{AQ} = \frac{1}{3} \quad (|||\triangle\text{s})$$
$$CD = \tfrac{1}{3}PQ$$
$$CD = 20 \text{ cm}$$
$$\text{But } AF = FC$$
$$AE = ED \quad \text{(mid-point th.)}$$
$$FE = \tfrac{1}{2}CD$$
$$FE = 20 \text{ cm}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 299Z 2. 29B2 3. 29B3 4. 29B4

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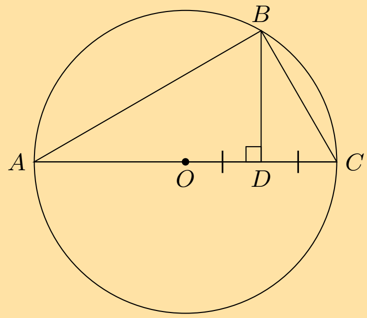
8.6 Pythagorean theorem

Exercise 8 – 9: Theorem of Pythagoras

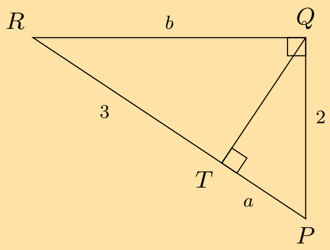
1. B is a point on circle with centre O. $BD \perp AC$ and D is the midpoint of radius OC. If the diameter of the circle is 24 cm, find BD. Leave answer in simplified surd form.



Solution:

$$
\begin{aligned}
ABC &= 90^\circ && (\angle \text{ in semi-circle})\\
BD &\perp AC\\
\therefore BD^2 &= AC.DC\\
AO &= OC && (\text{equal radii})\\
&= \tfrac{1}{2}AC\\
&= 12 \text{ cm}\\
\therefore DC &= 6 \text{ cm}\\
\therefore AD &= 18 \text{ cm}\\
BD^2 &= AD\ .\ DC && (\text{right-angled } \triangle\text{s})\\
BD^2 &= 18 \times 6\\
BD &= \sqrt{108}\\
&= \sqrt{36 \times 3}\\
&= 6\sqrt{3}\text{cm}
\end{aligned}
$$

2. In $\triangle PQR$, $RQ \perp QP$ and $QT \perp RP$. $PQ = 2$ units, $QR = b$ units, $RT = 3$ units and $TP = a$ units. Determine a and b, giving reasons.



Solution:

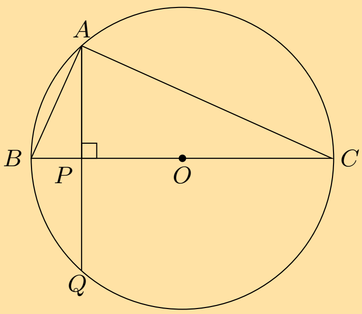
$$
\begin{aligned}
QP^2 &= PT\ .\ PR && (\text{right-angled } \triangle\text{s})\\
2^2 &= a(a+3)\\
4 &= a^2 + 3a\\
0 &= a^2 + 3a - 4\\
0 &= (a-1)(a+4)\\
\therefore a &= 1 \text{ or } a = -4
\end{aligned}
$$

Length must be positive $\therefore a = 1$ unit

$$
\begin{aligned}
QR^2 &= RT\ .\ RP && (\text{right-angled } \triangle\text{s})\\
b^2 &= 3(3+1)\\
&= 3(4)\\
&= 12\\
\therefore b &= \pm\sqrt{12}
\end{aligned}
$$

Length must be positive $\therefore b = 2\sqrt{3}$ units

3. Chord AQ of circle with centre O cuts BC at right angles at point P.



a) Why is $\triangle ABP \;|||\; \triangle CBA$?
Solution:

$$B\hat{A}C = 90^\circ \qquad (BC \text{ is diameter of circle } O)$$

In $\triangle ABP$ and $\triangle CBA$:

$$\begin{aligned} B\hat{P}A = B\hat{A}C &= 90^\circ && \text{(given)} \\ \hat{B} &= \hat{B} && \text{(common } \angle) \\ \therefore \triangle ABP \;&|||\; \triangle CBA && \text{(AAA)} \end{aligned}$$

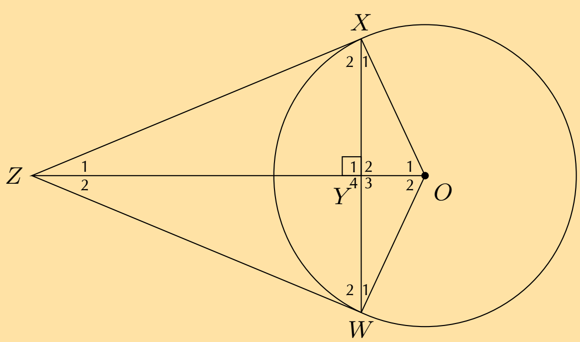
b) If $AB = \sqrt{6}$ units and $PO = 2$ units, calculate the radius of the circle.
Solution:
In $\triangle ABP$:

$$\begin{aligned} AP^2 &= BA^2 - BP^2 && \text{(Pythagoras)} \\ BP &= BO - PO \\ &= r - 2 && (BO = r, \text{ given } PO = 2) \\ AP^2 &= \left(\sqrt{6}\right)^2 - (r-2)^2 \\ &= 6 - (r-2)^2 \end{aligned}$$

$$\begin{aligned} AP^2 &= BP.PC && (B\hat{A}C = 90^\circ, AP \perp BC) \\ &= (r-2).PC \\ \text{And } PC &= PO + OC \\ &= 2 + r \\ \therefore AP^2 &= (r-2)(2+r) \end{aligned}$$

$$\begin{aligned} 6 - (r-2)^2 &= (r-2)(2+r) \\ 6 - r^2 + 4r - 4 &= r^2 - 4 \\ 2r^2 - 4r - 6 &= 0 \\ r^2 - 2r - 3 &= 0 \\ (r-3)(r+1) &= 0 \\ r &= 3 \text{ or } r = -1 \\ \therefore r &= 3 \text{ units} \end{aligned}$$

4. In the diagram below, XZ and WZ are tangents to the circle with centre O and $X\hat{Y}Z = 90^\circ$.



a) Show that $XY^2 = OY\ .\ YZ$.

Solution:

In $\triangle OXZ$:

$$X\hat{Y}Z = 90^\circ \quad \text{(given)}$$
$$O\hat{X}Z = 90^\circ \quad \text{(tangent } \perp \text{ radius)}$$
$$\therefore XY^2 = OY\ .\ YZ \quad \text{(right-angled } \triangle\text{s)}$$

b) Prove that $\frac{OY}{YZ} = \frac{OW^2}{WZ^2}$.

Solution:

In $\triangle OWZ$:

$$W\hat{Y}Z = 90^\circ \quad \text{(given)}$$
$$O\hat{W}Z = 90^\circ \quad \text{(tangent } \perp \text{ radius)}$$
$$\therefore WZ^2 = ZY\ .\ ZO \quad \text{(right-angled } \triangle\text{s)}$$
$$\text{And } WO^2 = OY\ .\ ZO \quad \text{(right-angled } \triangle\text{s)}$$
$$\therefore \frac{WO^2}{WZ^2} = \frac{OY\ .\ ZO}{ZY\ .\ ZO}$$
$$\frac{WO^2}{WZ^2} = \frac{OY}{ZY}$$
$$\therefore \frac{OY}{YZ} = \frac{OW^2}{WZ^2}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29B5 2. 29B6 3. 29B7 4. 29B8

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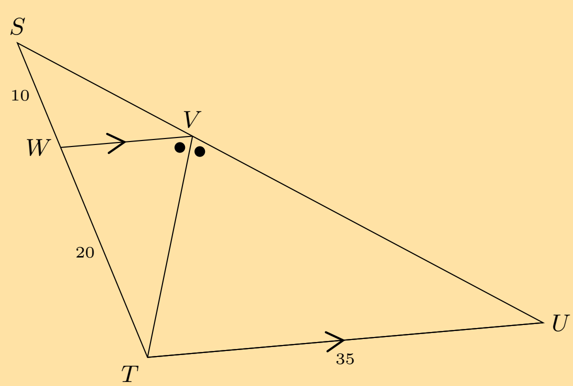
8.7 Summary

Exercise 8 – 10: End of chapter exercises

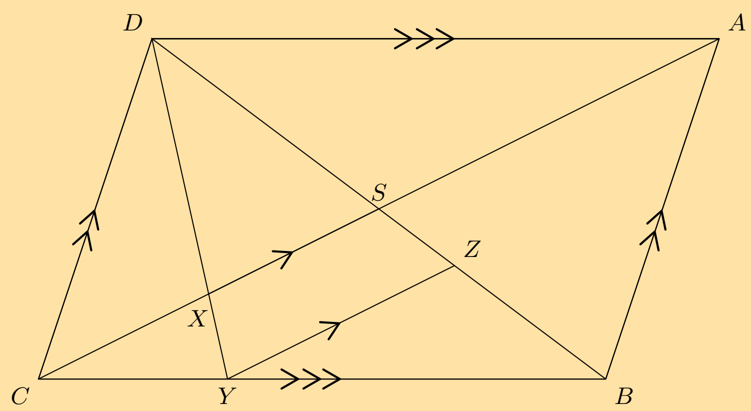
1. Calculate SV



Solution:

$$V\hat{T}U = W\hat{V}T \quad \text{(alt. } \angle\text{s}, WV \parallel TU)$$
$$\therefore TU = VU = 35 \quad \text{(isosceles } \triangle)$$
$$\frac{SW}{WT} = \frac{SV}{VU} \quad \text{(proportion Theorem)}$$
$$\therefore SV = \frac{SW.VU}{WT} = \frac{(10)(35)}{20} = 17{,}5 \text{ units}$$

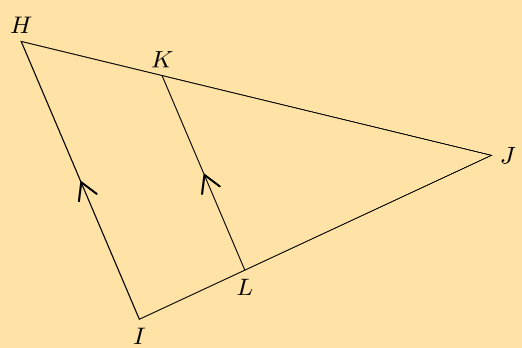
2. $\frac{CB}{YB} = \frac{3}{2}$. Find $\frac{DS}{SZ}$.



Solution:

$$\begin{aligned}
DABC \text{ is a parallelogram} & \quad (DA \parallel CB \text{ and } DC \parallel AB) \\
DS = SB & \quad \text{(diagonals bisect)} \\
\frac{SZ}{ZB} = \frac{CY}{YB} = \frac{3}{2} & \quad (CS \parallel YZ) \\
\frac{SZ}{SB} = \frac{CY}{CB} = \frac{3}{5} & \quad (CS \parallel YZ) \\
\therefore SZ = \frac{3}{5}SB & \\
\therefore \frac{DS}{SZ} = \frac{DS}{\frac{3}{5}SB} & \quad (DS = SB) \\
= \frac{5}{3} &
\end{aligned}$$

3. Using the following figure and lengths, find IJ and KJ (correct to one decimal place). $HI = 20$ m, $KL = 14$ m, $JL = 18$ m and $HJ = 32$ m.

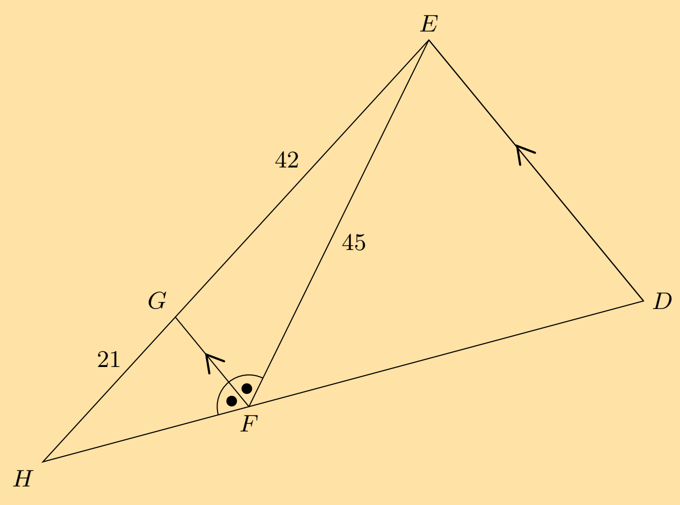


Solution:

$$\begin{aligned} \frac{IJ}{LJ} &= \frac{HI}{KL} && \text{(proportion Theorem)} \\ IJ &= \frac{HI}{KL}(LJ) \\ &= \frac{20}{14}(18) \\ &= \frac{180}{7} \\ &= 25{,}7 \text{ m} \end{aligned}$$

$$\begin{aligned} KJ &= \frac{LJ}{IJ}(HJ) \\ &= \frac{18}{25{,}7}(32) \\ &= 22{,}4 \text{ m} \end{aligned}$$

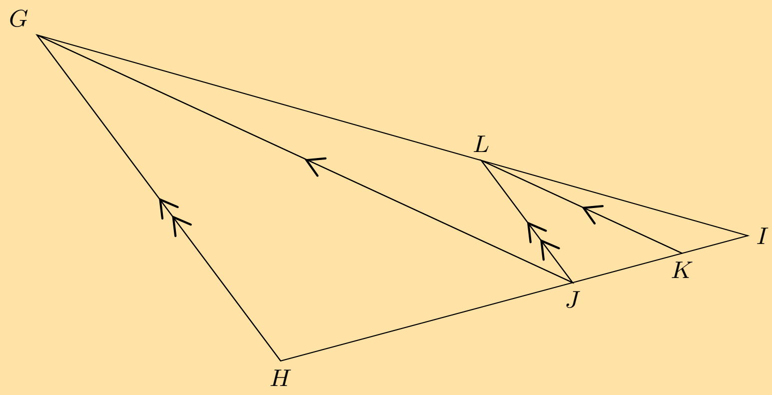
4. Find FH in the following figure.



Solution:

$$\begin{aligned} G\hat{F}H &= \hat{D} && \text{(corresp. } \angle\text{s}, GF \parallel ED) \\ G\hat{F}E &= F\hat{E}D && \text{(alt. } \angle\text{s}, GF \parallel ED) \\ \therefore F\hat{E}D &= \hat{D} \\ \therefore EF &= FD = 45 \text{ cm} \\ \frac{HF}{FD} &= \frac{21}{42} \\ &= \frac{1}{2} \\ \therefore HF &= \frac{1}{2}(45) \\ &= 23{,}5 \text{ cm} \end{aligned}$$

5. In $\triangle GHI$, $GH \parallel LJ$, $GJ \parallel LK$ and $\frac{JK}{KI} = \frac{5}{3}$. Determine $\frac{HJ}{KI}$.



Solution:

$$\begin{aligned} L\hat{I}J &= G\hat{I}H \\ J\hat{L}I &= H\hat{G}I && \text{(corresp. } \angle\text{s, } HG \parallel JL) \\ \therefore \triangle LIJ &\ ||| \ \triangle GIH && \text{(Equiangular } \triangle\text{s)} \end{aligned}$$

$$\begin{aligned} \frac{HJ}{JI} &= \frac{GL}{LI} && (\triangle LIJ\ |||\ \triangle GIH) \\ \text{and } \frac{GL}{LI} &= \frac{JK}{KI} && (\triangle LIK\ |||\ \triangle GIJ) \\ &= \frac{5}{3} \\ \therefore \frac{HJ}{JI} &= \frac{5}{3} \end{aligned}$$

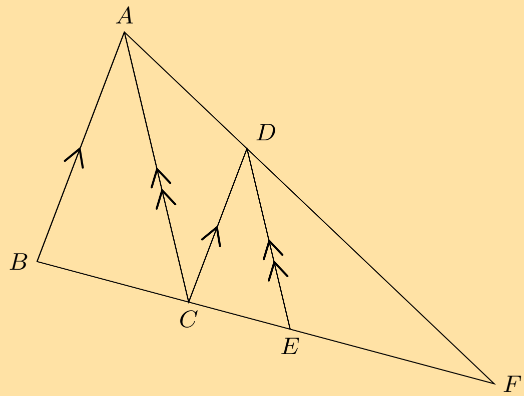
$$\frac{HJ}{KI} = \frac{HJ}{JI} \times \frac{JI}{KI}$$

$$\begin{aligned} JI &= JK + KI \\ &= \frac{5}{3}KI + KI \\ &= \frac{8}{3}KI \\ \frac{JI}{KI} &= \frac{8}{3} \end{aligned}$$

$$\begin{aligned} \frac{HJ}{KI} &= \frac{HJ}{JI} \times \frac{JI}{KI} \\ &= \frac{5}{3} \times \frac{8}{3} \\ &= \frac{40}{9} \end{aligned}$$

6. $BF = 25$ m, $AB = 13$ m, $AD = 9$ m, $DF = 18$ m.

   Calculate the lengths of BC, CF, CD, CE and EF, and find the ratio $\frac{DE}{AC}$.



Solution:

$$\frac{BC}{BF} = \frac{AD}{AF} = \frac{9}{27} = \frac{1}{3} \qquad (CD \parallel BA)$$
$$\therefore BC = \frac{1}{3} \times 25$$
$$= 8{,}3 \text{ m}$$

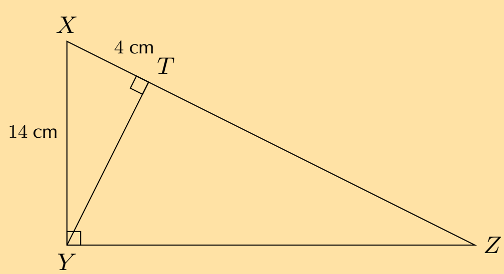
$$CF = BF - BC$$
$$= 25 - 8{,}3$$
$$= 16{,}7 \text{ m}$$

$$\frac{CD}{AB} = \frac{DF}{AF} \qquad (CD \parallel BA)$$
$$CD = \frac{DF}{AF} \times AB$$
$$= \frac{18}{27} \times 13$$
$$= 8{,}7 \text{ m}$$

$$\frac{CE}{CF} = \frac{AD}{AF} \qquad (DE \parallel AC)$$
$$CE = \frac{AD}{AF} \times CF$$
$$= \frac{9}{27} \times 16{,}7$$
$$= 5{,}6 \text{ m}$$

$$EF = BF - (BC + CE)$$
$$= 25 - (8{,}3 - 5{,}6)$$
$$= 11{,}1 \text{ m}$$

7. In $\triangle XYZ$, $X\hat{Y}Z = 90°$ and $YT \perp XZ$. If $XY = 14$ cm and $XT = 4$ cm, determine XZ and YZ (correct to two decimal places).



Solution:

Use the theorem of Pythagoras to determine YT:

$$\text{In } \triangle XTY, \quad YT^2 = XY^2 - XT^2 \qquad \text{(Pythagoras)}$$
$$= 14^2 - 4^2$$
$$= 196 - 16$$
$$\therefore YT = \sqrt{180}$$
$$= \sqrt{36 \times 5} = 6\sqrt{5} \text{ cm}$$

Use proportionality to determine XZ and YZ:

$$\begin{aligned} X\hat{Y}Z &= 90° && \text{(given)} \\ YT &\perp XZ && \text{(given)} \\ \therefore \triangle XYT &\ ||| \ \triangle YZT \ ||| \ \triangle XZY && \text{(right-angled } \triangle\text{s)} \end{aligned}$$

$$\therefore \frac{YT}{TZ} = \frac{XT}{YT} \qquad (\triangle YZT \ ||| \ \triangle XYT)$$

$$\therefore YT^2 = TZ \ . \ XT$$

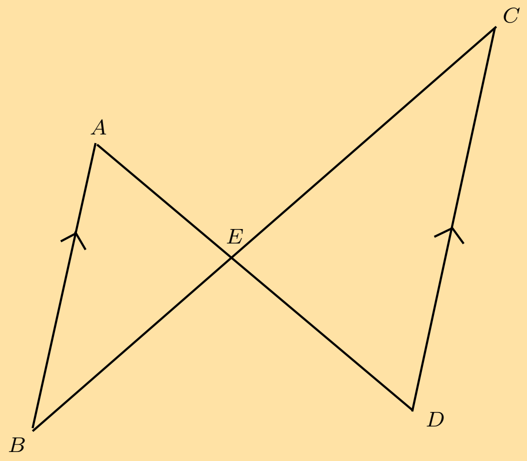
$$\left(6\sqrt{5}\right)^2 = TZ \ . \ 4$$

$$\therefore TZ = 45$$

$$\begin{aligned} \text{And } XZ &= XT + TZ \\ &= 4 + 45 \\ &= 49 \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{In } \triangle XYZ, \quad YZ^2 &= XZ^2 - XY^2 && \text{(Pythagoras)} \\ &= 49^2 - 14^2 \\ \therefore YZ &= \sqrt{2205} \\ &= 46{,}96 \text{ cm} \end{aligned}$$

8. Given the following figure with the following lengths, find AE, EC and BE. $BC = 15$ cm, $AB = 4$ cm, $CD = 18$ cm, and $ED = 9$ cm.



Solution:

$$\begin{aligned} B\hat{A}E &= C\hat{D}E && \text{(alt. } \angle\text{s, } AB \parallel CD) \\ A\hat{B}E &= D\hat{C}E && \text{(alt. } \angle\text{s, } AB \parallel CD) \\ A\hat{E}B &= D\hat{E}C && \text{(vert. opp. } \angle\text{s)} \\ \therefore \triangle AEB &\ ||| \ \triangle DEC && \text{(AAA)} \\ \therefore \tfrac{AE}{DE} &= \tfrac{AB}{DC} = \tfrac{4}{18} = \tfrac{2}{9} && (\triangle AEB \ ||| \ \triangle DEC) \end{aligned}$$

$$\begin{aligned} AE &= \frac{2}{9} DE \\ &= \frac{2}{9}(9) \\ &= 2 \text{ cm} \end{aligned}$$

$$\frac{EC}{BC} = \frac{ED}{AD} = \frac{9}{11} \qquad (AB \parallel CD)$$

$$\begin{aligned} EC &= \frac{ED}{AD}(BC) \\ &= \frac{9}{11}(15) = 12{,}3 \text{ cm} \end{aligned}$$

$$\frac{BE}{BC} = \frac{AE}{AD} = \frac{2}{11} \qquad (AB \parallel CD)$$

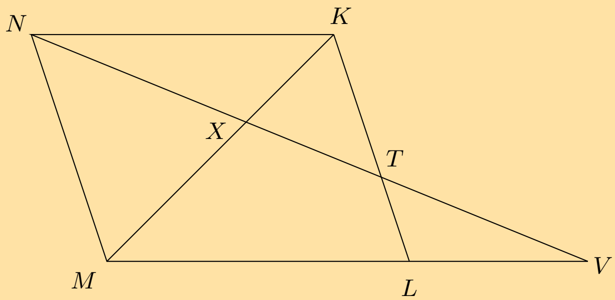
$$BE = \frac{AE}{AD}(BC)$$

$$= \frac{2}{11}(15)$$

$$= 2{,}7 \text{ cm}$$

9. $NKLM$ is a parallelogram with T on KL.

   NT produced meets ML produced at V. NT intercepts MK at X.



   a) Prove that $\frac{XT}{NX} = \frac{XK}{MX}$.

   Solution:

   In $\triangle TXK$ and $\triangle NXM$:

$$\begin{aligned} X\hat{T}K &= X\hat{N}M && \text{(alt. } \angle\text{s, } NK \parallel MV) \\ N\hat{X}M &= T\hat{X}K && \text{(vert. opp. } \angle\text{s)} \\ \therefore \triangle TXK &\ ||| \ \triangle NXM && \text{(AAA)} \\ \therefore \frac{TX}{NX} &= \frac{XK}{XM} \end{aligned}$$

   b) Prove $\triangle VXM \ ||| \ \triangle NXK$.

   Solution:

   In $\triangle VXM$ and $\triangle NXK$:

$$\begin{aligned} \hat{V} &= X\hat{N}K && \text{(alt. } \angle\text{s, } NK \parallel MV) \\ M\hat{X}V &= K\hat{X}N && \text{(vert. opp. } \angle\text{s)} \\ \therefore \triangle VXM &\ ||| \ \triangle NXK && \text{(AAA)} \end{aligned}$$

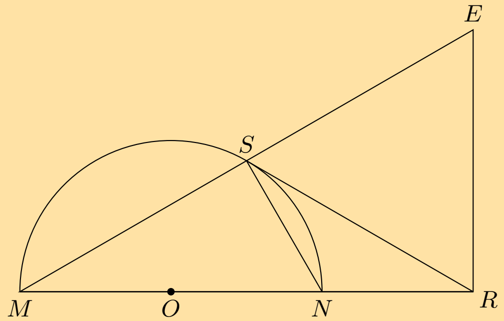
   c) If $XT = 3$ cm and $TV = 4$ cm, calculate NX.

   Solution:

$$\begin{aligned} \frac{VX}{NX} &= \frac{XM}{XK} && (\triangle VXM \ ||| \ \triangle NXK\text{, proved in (b))} \\ \text{But } \frac{XM}{XK} &= \frac{NX}{TX} && \text{(proved in (a))} \\ \therefore \frac{VX}{NX} &= \frac{NX}{TX} \\ NX^2 &= VX.TX \\ NX^2 &= 3 \times 4 \\ NX &= \sqrt{12} \\ &= 2\sqrt{3} \text{ cm} \end{aligned}$$

10. MN is a diameter of circle O. MN is produced to R so that $MN = 2NR$.

    RS is a tangent to the circle and $ER \perp MR$. MS produced meets RE at E.



Prove that:

a) $SNRE$ is a cyclic quadrilateral

Solution:

$$M\hat{S}N = 90^\circ \quad (\angle \text{ in semi-circle})$$
$$N\hat{R}E = 90^\circ \quad (\text{given})$$
$$\therefore SNRE \text{ is a cyclic quad.} \quad (\text{ext. } \angle = \text{ opp. int. } \angle)$$

b) $RS = RE$

Solution:

$$N\hat{S}R = \hat{M} = x \quad (\text{tangent/chord})$$
$$\therefore E\hat{S}R = 90^\circ - x$$
$$\hat{E} = 90^\circ - x \quad (MRE = 90^\circ, \ \hat{M} = x)$$
$$\therefore E\hat{S}R = \hat{E}$$
$$\therefore RS = RE \quad (\text{isos. } \triangle)$$

c) $\triangle MSN \ ||| \ \triangle MRE$

Solution:

In $\triangle MSN$ and $\triangle MRE$:

$$\hat{M} = \hat{M}$$
$$M\hat{S}N = 90^\circ \quad (\angle \text{ in semi-circle})$$
$$M\hat{R}E = 90^\circ \quad (\text{given})$$
$$\therefore M\hat{S}N = M\hat{R}E$$
$$\therefore \triangle MSN \ ||| \ \triangle MRE \quad (\text{AAA})$$

d) $\triangle RSN \ ||| \ \triangle RMS$

Solution:

In $\triangle RSN$ and $\triangle RMS$:

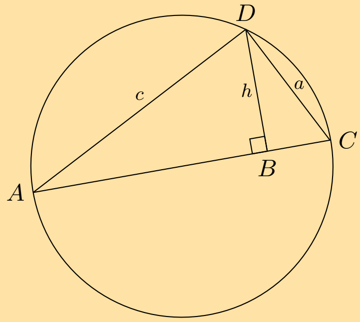
$$\hat{R} = \hat{R} \quad (\text{common } \angle)$$
$$R\hat{S}N = \hat{M} \quad (\text{tangent/chord})$$
$$\therefore \triangle RSN \ ||| \ \triangle RMS \quad (\text{AAA})$$

e) $RE^2 = RN.RM$

Solution:

$$\frac{RS}{RN} = \frac{RM}{RS}$$
$$RS^2 = RN.RM$$
$$\text{But } RS = RE$$
$$RE^2 = RN.RM$$

11. AC is a diameter of circle ADC. $DB \perp AC$.
$AC = d$, $AD = c$, $DC = a$ and $DB = h$.



a) Prove that $h = \dfrac{ac}{d}$.

Solution:

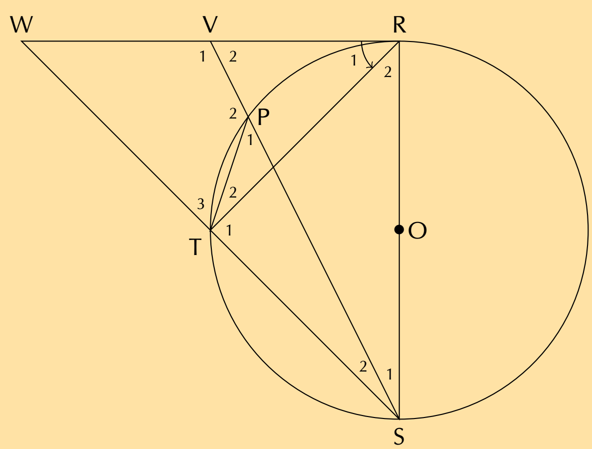
$$\begin{aligned}
\triangle ADB \quad ||| \ \triangle DCB \ ||| \ \triangle ACD \qquad & (A\hat{D}C = 90^\circ, DB \perp AC)\\
\therefore \frac{DB}{AD} &= \frac{CD}{AC}\\
\therefore \frac{h}{c} &= \frac{a}{d}\\
\therefore h &= \frac{ac}{d}
\end{aligned}$$

b) Hence, deduce that $\dfrac{1}{h^2} = \dfrac{1}{a^2} + \dfrac{1}{c^2}$.

Solution:

$$\begin{aligned}
h^2 &= \frac{a^2c^2}{d^2}\\
\text{But } d^2 &= a^2 + c^2 \qquad \text{(In } \triangle ADC, \hat{D} = 90^\circ, \text{ Pythagoras)}\\
\therefore h^2 &= \frac{a^2c^2}{a^2+c^2}\\
\therefore \frac{1}{h^2} &= \frac{a^2+c^2}{a^2c^2}\\
\frac{1}{h^2} &= \frac{a^2}{a^2c^2} + \frac{c^2}{a^2c^2}\\
\frac{1}{h^2} &= \frac{1}{c^2} + \frac{1}{a^2}
\end{aligned}$$

12. RS is a diameter of the circle with centre O. Chord ST is produced to W. Chord SP produced meets the tangent RW at V. $\hat{R}_1 = 50^\circ$.
[NCS, Paper 3, November 2011]



a) Calculate the size of $W\hat{R}S$.

Solution:

$W\hat{R}S = 90°$ (tangent $\perp$ radius)

b) Find $\hat{W}$.

Solution:

$$R\hat{S}T = 50° \quad \text{(tangent chord th.)}$$
$$\hat{W} = 40° \quad (\angle \text{ sum } \triangle)$$

OR

$$\hat{T}_1 = 90° \quad (\angle\text{s in semi-circle})$$
$$\hat{W} + \hat{R}_1 = \hat{T}_1 \quad (\text{ ext. } \angle\triangle)$$
$$\hat{W} = 40°$$

c) Determine the size of $\hat{P}_1$.

Solution:

$$\hat{R}_2 = 40° \quad \text{(tangent } \perp \text{ radius)}$$
$$\hat{P}_1 = 40° \quad (\angle\text{s in same seg.})$$

d) Prove that $\hat{V}_1 = P\hat{T}S$.

Solution:

$$\hat{P}_1 = \hat{W} \quad (= 40°)$$
$$\text{WVPT is a cyclic quadrilateral} \quad (\text{ext. } \angle = \text{ int. opp.})$$
$$\hat{V}_1 = P\hat{T}S \quad (\text{ext. } \angle \text{ cyclic quad.})$$

OR

$$\hat{T}_1 = 90° \quad (\angle\text{s in semi-circle})$$
$$P\hat{T}S = 90° + \hat{T}_2$$
$$\hat{T}_2 = \hat{S}_1 \quad (\angle\text{s in same seg.})$$
$$P\hat{T}S = 90° + \hat{S}_1 \quad (\text{ext. } \angle\triangle)$$
$$\hat{V}_1 = P\hat{T}S$$

OR

$$\hat{P}_2 = 140° \quad (\angle\text{s on str. line})$$
$$\hat{W} + \hat{P}_2 = 180°$$
$$\text{WVPT is a cyclic quadrilateral} \quad (\text{opp. } \angle\text{s suppl})$$
$$\hat{V}_1 = P\hat{T}S \quad (\text{ext } \angle \text{ cyclic quad})$$

OR

$$\hat{V}_1 = \hat{R}_1 + \hat{R}_2 + \hat{S}_1 \quad (\text{ext. } \angle\triangle)$$
$$\hat{V}_1 = 90° + \hat{S}_1$$
$$P\hat{T}S = 90° + \hat{T}_2$$
$$\text{but } \hat{T}_2 = \hat{S}_1 \quad (\angle\text{s in same seg.})$$
$$\hat{V}_1 = P\hat{T}S$$

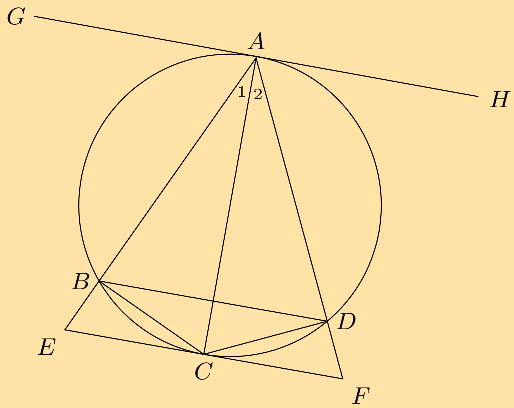
OR

In $\triangle PTS$ and $\triangle WVS$

$$\hat{P}_1 = \hat{W} \quad (40°)$$
$$\hat{S}_2 \text{ is common}$$
$$\hat{V}_1 = P\hat{T}S \quad (\angle \text{ sum } \triangle)$$

13. $ABCD$ is a cyclic quadrilateral and $BC = CD$.

ECF is a tangent to the circle at C. ABE and ADF are straight lines.



Prove:

a) AC bisects $E\hat{A}F$

Solution:

$$BC = CD \quad \text{(given)}$$
$$\therefore B\hat{A}C = D\hat{A}C \quad (\angle\text{s on equal chords})$$

b) $BD \parallel EF$

Solution:

$$D\hat{C}F = \hat{A}_2 \quad \text{(tangent/chord)}$$
$$B\hat{D}C = \hat{A}_1 \quad (\angle\text{s on same chord})$$
$$\hat{A}_1 = \hat{A}_2 \quad \text{(proved in (a))}$$
$$\therefore B\hat{D}C = D\hat{C}F$$
$$\therefore BD \parallel EF \quad (\text{alt. } \angle\text{s are equal})$$

c) $\triangle ADC \; ||| \; \triangle CBE$

Solution:

In $\triangle ADC$ and $\triangle CBE$:

$$\hat{A}_2 = D\hat{B}C \quad (\angle\text{s on chord } CD)$$
$$= B\hat{C}E \quad (\text{alt. } \angle\text{s, } BD \parallel EF)$$
$$A\hat{D}C = E\hat{B}C \quad (\text{ext. } \angle \text{ of a cyclic quad})$$
$$\therefore \triangle ADC \; ||| \; \triangle CBE \quad \text{(AAA)}$$

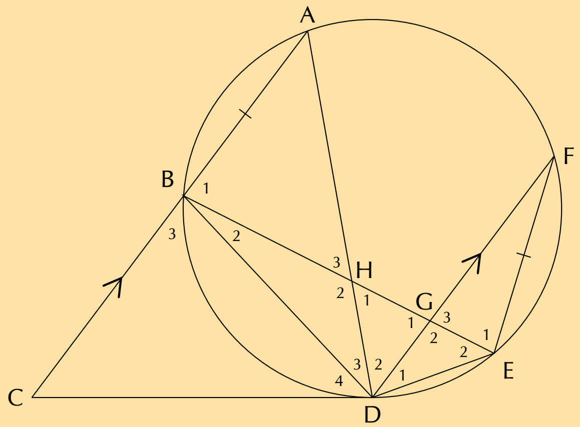
d) $DC^2 = AD.BE$

Solution:

$$\frac{DC}{AD} = \frac{BE}{BC} \quad (\triangle ADC \; ||| \; \triangle CBE)$$
$$\text{But } DC = BC \quad \text{(given)}$$
$$\therefore \frac{DC}{AD} = \frac{BE}{DC}$$
$$\therefore DC^2 = AD.BE$$

14. CD is a tangent to circle $ABDEF$ at D. Chord AB is produced to C. Chord BE cuts chord AD in H and chord FD in G. $AC \parallel FD$ and $E = AB$. Let $D_4 = x$ and $D_1 = y$.

[NCS, Paper 3, November 2011]



a) Determine THREE other angles that are each equal to x.

Solution:

$$\begin{aligned} \hat{A} = \hat{D}_4 = x & \quad \text{(tangent chord th.)} \\ \hat{E}_2 = x & \quad \text{(tangent chord th.)} \\ \hat{D}_2 = \hat{A} = x & \quad \text{(alt. } \angle\text{s, } CA \parallel DF) \end{aligned}$$

b) Prove that $\triangle$BHD ||| $\triangle$FED.

Solution:

In $\triangle BHD$ and $\triangle FED$

i. $\hat{B}_2 = \hat{F}$ ($\angle$s in same seg.)

ii. $\hat{D}_3 = \hat{D}_1$ (chord subtends $= \angle$s)

$\triangle BHD \; ||| \; \triangle FED$ ($\angle\angle\angle$)

c) Hence, or otherwise, prove that $AB \,.\, BD = FD \,.\, BH$.

Solution:

$$\begin{aligned} \frac{FE}{BH} &= \frac{FD}{BD} & \text{(||| } \triangle\text{s)} \\ \text{But } FE &= AB & \text{(given)} \\ \frac{AB}{BH} &= \frac{FD}{BD} \\ AB \,.\, BD &= FD \,.\, BH \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29B9 2. 29BB 3. 29BC 4. 29BD 5. 29BF 6. 29BG
7. 29BH 8. 29BJ 9. 29BK 10. 29BM 11. 29BN 12. 29BP
13. 29BQ 14. 29BR

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CHAPTER 9

Statistics

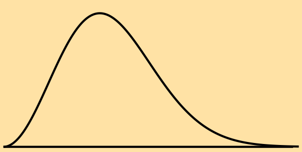
- It is easy for learners to be intimidated by the formulae in this chapter. Slowly go through the meaning of each term in the formulae and ensure that learners are familiar with summation notation.
- Use by hand calculations to aid the understanding of key concepts.
- Calculator skills are very important in this chapter. Methods for SHARP and CASIO calculators are shown but practical demonstration may be required.
- The formula for population variance and population standard deviation are used and not sample variance and sample standard deviation.
- Calculation of the correlation coefficient requires the use of the sample standard deviation, but the concepts of sample and population are beyond the scope of CAPS. The formula used for the correlation coefficient in this chapter is $r = b\frac{\sigma_x}{\sigma_y}$ so as not to confuse learners. The actual formula is $r = b\frac{S_x}{S_y}$, however as this is a ratio, the difference in denominators between the population standard deviation and sample deviation will cancel. Note that there are other formulae for the correlation coefficient but these should be avoided as usage of the population standard deviation will result in erroneous results.
- Discuss the misuse of statistics in the real world and encourage awareness.

9.1 Revision

Exercise 9 – 1: Revision

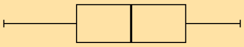
1. State whether each of the following data sets are symmetric, skewed right or skewed left.

   a) A data set with this distribution:

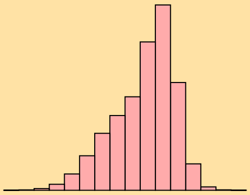
   Solution: skewed right

   b) A data set with this box and whisker plot:

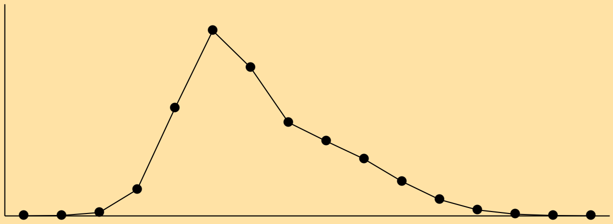
   Solution: symmetric

   c) A data set with this histogram:

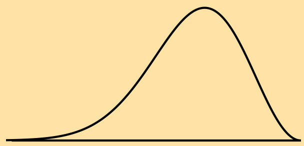
   Solution: skewed left

   d) A data set with this frequency polygon:

Solution: skewed right

e) A data set with this distribution:



Solution: skewed left

f) The following data set:

$105\ ;\ 44\ ;\ 94\ ;\ 149\ ;\ 83\ ;\ 178\ ;\ -4\ ;\ 112\ ;\ 50\ ;\ 188$

Solution:

The statistics of the data set are

- mean: $99{,}9$;
- first quartile: $66{,}5$;
- median: $99{,}5$;
- third quartile: $130{,}5$.

Note that we get contradicting indications from the different ways of determining whether the data is skewed right or left.

- The mean is slightly greater than the median. This would indicate that the data set is skewed right.
- The median is slightly closer to the third quartile than the first quartile. This would indicate that the data set is skewed left.

Since these differences are so small and since they contradict each other, we conclude that the data set is symmetric.

2. For the following data sets:

- Determine the mean and five number summary.
- Draw the box and whisker plot.
- Determine the skewness of the data.

a) $40\ ;\ 45\ ;\ 12\ ;\ 6\ ;\ 9\ ;\ 16\ ;\ 11\ ;\ 7\ ;\ 35\ ;\ 7\ ;\ 31\ ;\ 3$

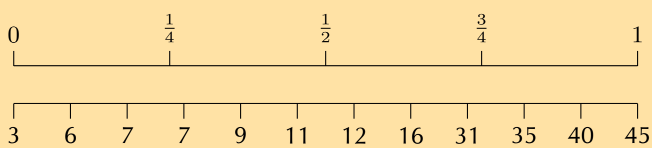
Solution:

The mean is $\bar{x} = \frac{222}{12} = 18{,}5$.

To determine the five number summary, we order the data:

$$3\ ;\ 6\ ;\ 7\ ;\ 7\ ;\ 9\ ;\ 11\ ;\ 12\ ;\ 16\ ;\ 31\ ;\ 35\ ;\ 40\ ;\ 45$$

We use the diagram below (or the formulae) to find at, or between, which values the quartiles lie.



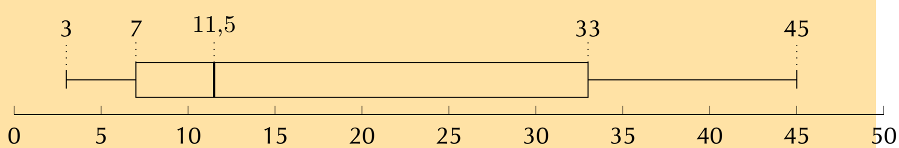
The minimum is 3 and the maximum is 45.

For the first quartile, the position is between the third and fourth values. Since both these values are equal to 7, the first quartile is 7.

For the median (second quartile), the position is halfway between the sixth and seventh values. The sixth value is 11 and the seventh value is 12, which means that the median is $\frac{11+12}{2} = 11{,}5$.

For the third quartile, the position is between the ninth and tenth values. Therefore the third quartile is $\frac{31+35}{2} = 33$.

Therefore, the five number summary is (3; 7; 11,5; 33; 45)



$$\text{mean} - \text{median} = 18{,}5 - 11{,}5 = 7$$

Mean $>$ median, therefore the data is skewed right.

b) $65 ; 100 ; 99 ; 21 ; 8 ; 27 ; 21 ; 31 ; 33 ; 31 ; 38 ; 16$

Solution:

The mean is $\bar{x} = \frac{490}{12} = 40{,}83$.

To determine the five number summary, we order the data:

$$8\ ;\ 16\ ;\ 21\ ;\ 21\ ;\ 27\ ;\ 31\ ;\ 31\ ;\ 33\ ;\ 38\ ;\ 65\ ;\ 99\ ;\ 100$$

We use the formulae (or a diagram) to find at, or between, which values the quartiles lie.

$$\text{Location of } Q_1 = \frac{1}{4}(n-1)+1 = \frac{1}{4}(12-1)+1 = 3{,}75$$
$$\text{Location of } Q_2 = \frac{1}{2}(n-1)+1 = \frac{1}{2}(12-1)+1 = 6{,}5$$
$$\text{Location of } Q_3 = \frac{3}{4}(n-1)+1 = \frac{3}{4}(12-1)+1 = 9{,}25$$

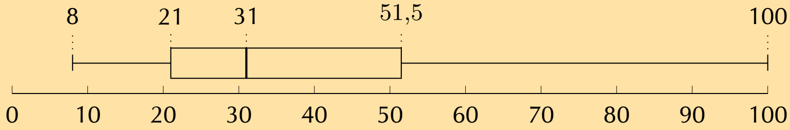
The minimum is 8 and the maximum is 100.

For the first quartile, the position is between the third and fourth values. Since both these values are equal to 21, the first quartile is 21.

For the median (second quartile), the position is halfway between the sixth and seventh values. The sixth value and seventh value are 31, which means that the median is 31.

For the third quartile, the position is between the ninth and tenth values. Therefore the third quartile is $\frac{38+65}{2} = 51{,}5$.

Therefore, the five number summary is (8; 21; 31; 51,5; 100)



$$\text{mean} - \text{median} = 40{,}83 - 31 = 9{,}83$$

Mean $>$ median, therefore the data is skewed right.

c) $65 ; 57 ; 77 ; 92 ; 77 ; 58 ; 90 ; 46 ; 11 ; 81$

Solution:

The mean is $\bar{x} = \frac{654}{10} = 65{,}4$.

To determine the five number summary, we order the data:

$$11\ ;\ 46\ ;\ 57\ ;\ 58\ ;\ 65\ ;\ 77\ ;\ 77\ ;\ 81\ ;\ 90\ ;\ 92$$

We use the formulae (or a diagram) to find at, or between, which values the quartiles lie.

$$\text{Location of } Q_1 = \frac{1}{4}(n-1)+1 = \frac{1}{4}(10-1)+1 = 3{,}25$$
$$\text{Location of } Q_2 = \frac{1}{2}(n-1)+1 = \frac{1}{2}(10-1)+1 = 5{,}5$$
$$\text{Location of } Q_3 = \frac{3}{4}(n-1)+1 = \frac{3}{4}(10-1)+1 = 7{,}75$$

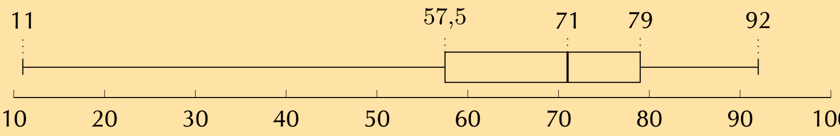
The minimum is 11 and the maximum is 92.

For the first quartile, the position is between the third and fourth values. Since the third is 57 and the fourth is 58, the first quartile is $\frac{57+58}{2} = 57{,}5$.

For the median (second quartile), the position is halfway between the fifth and sixth values. The sixth value is 65 and the seventh value is 77, therefore the median is $\frac{65+77}{2} = 71$.

For the third quartile, the position is between the seventh and eighth values. Therefore the third quartile is $\frac{77+81}{2} = 79$.

Therefore, the five number summary is (11; 57,5; 77; 79; 92)



$$\text{mean} - \text{median} = 65{,}4 - 71 = -5{,}6$$

Mean < median, therefore the data is skewed left.

d) $1 ; 99 ; 76 ; 76 ; 50 ; 74 ; 83 ; 91 ; 41 ; 17 ; 33$

Solution:

The mean is $\bar{x} = \frac{641}{11} = 58{,}27$.

To determine the five number summary, we order the data:

$$1 \ ; \ 17 \ ; \ 33 \ ; \ 41 \ ; \ 50 \ ; \ 74 \ ; \ 76 \ ; \ 76 \ ; \ 83 \ ; \ 91 \ ; \ 99$$

We use the formulae (or a diagram) to find at, or between, which values the quartiles lie.

$$\text{Location of } Q_1 = \frac{1}{4}(n-1)+1 = \frac{1}{4}(11-1)+1 = 3{,}5$$

$$\text{Location of } Q_2 = \frac{1}{2}(n-1)+1 = \frac{1}{2}(11-1)+1 = 6$$

$$\text{Location of } Q_3 = \frac{3}{4}(n-1)+1 = \frac{3}{4}(11-1)+1 = 8{,}5$$

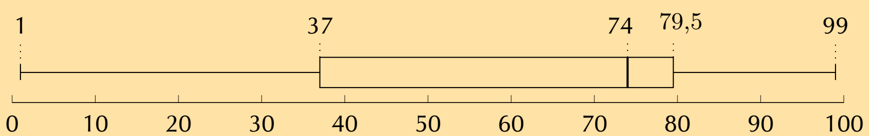
The minimum is 1 and the maximum is 99.

For the first quartile, the position is between the third and fourth values. The third value is 33 and the fourth value is 41, therefore the first quartile is $\frac{33+41}{2} = 37$.

The second quartile (median) is at the sixth position. The sixth value is 74.

For the third quartile, the position is between the eighth and ninth values. Therefore the third quartile is $\frac{76+83}{2} = 79{,}5$.

Therefore, the five number summary is (1; 37; 74; 79,5; 99)



$$\text{mean} - \text{median} = 58{,}27 - 74 = -15{,}73$$

Mean < median, therefore the data is skewed left.

e) $0{,}5; -0{,}9 ; -1{,}8 ; 3 ; -0{,}2 ; -5{,}2 ; -1{,}8 ; 0{,}1 ; -1{,}7 ; -2 ; 2{,}2 ; 0{,}5; -0{,}5$

Solution:

The mean is $\bar{x} = \frac{-7{,}8}{13} = -0{,}6$.

To determine the five number summary, we order the data:

$$-5{,}2; -2; -1{,}8; -1{,}8; -1{,}7; -0{,}9; -0{,}5; -0{,}2 \ ; \ 0{,}1 \ ; \ 0{,}5 \ ; \ 0{,}5 \ ; \ 2{,}2 \ ; \ 3$$

We use the formulae (or a diagram) to find at, or between, which values the quartiles lie.

$$\text{Location of } Q_1 = \frac{1}{4}(n-1)+1 = \frac{1}{4}(13-1)+1 = 4$$

$$\text{Location of } Q_2 = \frac{1}{2}(n-1)+1 = \frac{1}{2}(13-1)+1 = 7$$

$$\text{Location of } Q_3 = \frac{3}{4}(n-1)+1 = \frac{3}{4}(13-1)+1 = 10$$

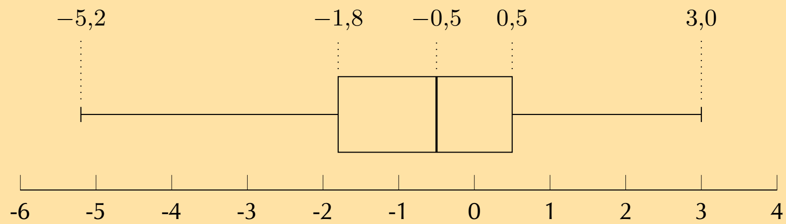
The minimum is $-5{,}2$ and the maximum is 3.

The first quartile is at the fourth position. Therefore, the first quartile is $-1{,}8$.

The median (second quartile) is at the seventh position. The seventh value is $-0{,}5$.

The third quartile is at the tenth position. Therefore the third quartile is $0{,}5$.

Therefore, the five number summary is $(-5{,}2; -1{,}8; -0{,}5; 0{,}5; 3)$



$$\text{mean} - \text{median} = -0{,}6 - (-0{,}5) = -0{,}1$$

Mean $<$ median, however this difference very small. Therefore the data is close to symmetric/slightly skewed left.

f) $86 ; 64 ; 25 ; 71 ; 54 ; 44 ; 97 ; 31 ; 78 ; 46 ; 60 ; 86$

Solution:

The mean is $\bar{x} = \frac{742}{12} = 61{,}83$.

To determine the five number summary, we order the data:

$$25 \ ; \ 31 \ ; \ 44 \ ; \ 46 \ ; \ 54 \ ; \ 60 \ ; \ 64 \ ; \ 71 \ ; \ 78 \ ; \ 86 \ ; \ 86 \ ; \ 97$$

We use the formulae (or a diagram) to find at, or between, which values the quartiles lie.

$$\text{Location of } Q_1 = \frac{1}{4}(n-1)+1 = \frac{1}{4}(12-1)+1 = 3{,}75$$
$$\text{Location of } Q_2 = \frac{1}{2}(n-1)+1 = \frac{1}{2}(12-1)+1 = 6{,}5$$
$$\text{Location of } Q_3 = \frac{3}{4}(n-1)+1 = \frac{3}{4}(12-1)+1 = 9{,}25$$

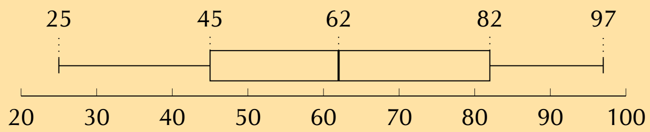
The minimum is 25 and the maximum is 97.

For the first quartile, the position is between the third and fourth values. The third value is 44 and the fourth value is 46, which means that the first quartile is $\frac{44+46}{2} = 45$.

For the median (second quartile), the position is halfway between the sixth and seventh values. The sixth value is 60 and the seventh value is 64, which means that the median is $\frac{60+64}{2} = 62$.

For the third quartile, the position is between the ninth and tenth values. Therefore the third quartile is $\frac{78+86}{2} = 82$.

Therefore, the five number summary is (25; 45; 62; 82; 97)



$$\text{mean} - \text{median} = 61{,}83 - 62 = -0{,}17$$

Mean $\approx$ median, therefore the data is symmetric.

3. For the following data sets:

- Determine the mean.
- Use a table to determine the variance and the standard deviation.
- Determine the percentage of data points within one standard deviation of the mean. Round your answer to the nearest percentage point.

a) $\{9{,}1; 0{,}2; 2{,}8; 2{,}0; 10{,}0; 5{,}8; 9{,}3; 8{,}0\}$

Solution:

The formula for the mean is

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$
$$\therefore \bar{x} = \frac{47{,}2}{8}$$
$$= 5{,}90$$

The formula for the variance is

$$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n}$$

We first subtract the mean from each data point and then square the result.

x_i	9,1	0,2	2,8	2,0	10,0	5,8	9,3	8,0
$x_i - \overline{x}$	3,2	−5,7	−3,1	−3,9	4,1	−0,1	3,4	2,1
$(x_i - \overline{x})^2$	10,24	32,49	9,61	15,21	16,81	0,01	11,56	4,41

The variance is the sum of the last row in this table divided by 8, so $\sigma^2 = \frac{47{,}2}{8} = 12{,}54$. The standard deviation is the square root of the variance, therefore $\sigma = \sqrt{12.54} = \pm 3{,}54$.

The interval containing all values that are one standard deviation from the mean is $[5{,}90 - 3{,}54; 5{,}90 + 3{,}54] = [9{,}44; 2{,}36]$. We are asked how many values are **within** than one standard deviation from the mean, meaning **inside** the interval. There are 5 values from the data set within the interval, which is $\frac{5}{8} \times 100 = 63\%$ of the data points.

b) {9; 5; 1; 3; 3; 5; 7; 4; 10; 8}

Solution:

The formula for the mean is

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$
$$\therefore \bar{x} = \frac{55}{10}$$
$$= 5{,}5$$

The formula for the variance is

$$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n}$$

We first subtract the mean from each data point and then square the result.

x_i	9	5	1	3	3	5	7	4	10	8
$x_i - \overline{x}$	3,5	−0,5	−4,5	−2,5	−2,5	−0,5	1,5	−1,5	4,5	2,5
$(x_i - \overline{x})^2$	12,25	0,25	20,25	6,25	6,25	0,25	2,25	2,25	20,25	6,25

The variance is the sum of the last row in this table divided by 10, so $\sigma^2 = \frac{76{,}5}{10} = 7{,}65$. The standard deviation is the square root of the variance, therefore $\sigma = \sqrt{7{,}65} = \pm 2{,}77$.

The interval containing all values that are one standard deviation from the mean is $[5{,}5 - 2{,}77; 5{,}5 + 2{,}77] = [2{,}73; 8{,}27]$. We are asked how many values are **within** than one standard deviation from the mean, meaning **inside** the interval. There are 7 values from the data set within the interval, which is $\frac{7}{10} \times 100 = 70\%$ of the data points.

c) {81; 22; 63; 12; 100; 28; 54; 26; 50; 44; 4; 32}

Solution:
The formula for the mean is

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$
$$\therefore \bar{x} = \frac{516}{12}$$
$$= 43$$

The formula for the variance is

$$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n}$$

We first subtract the mean from each data point and then square the result.

x_i	81	22	63	12	100	28	54	26	50	44	4	32
$x_i - \overline{x}$	38	-21	20	-31	57	-15	11	-17	7	1	-39	-11
$(x_i - \overline{x})^2$	1444	441	400	961	3249	225	121	289	49	1	1521	121

The variance is the sum of the last row in this table divided by 12, so $\sigma^2 = \frac{8822}{12} = 735{,}17$. The standard deviation is the square root of the variance, therefore $\sigma = \sqrt{735.17} = \pm 27{,}11$.
The interval containing all values that are one standard deviation from the mean is $[43 - 27{,}11; 43 + 27{,}11] = [15{,}89; 70{,}11]$. We are asked how many values are **within** than one standard deviation from the mean, meaning **inside** the interval. There are 8 values from the data set within the interval, which is $\frac{8}{12} \times 100 = 67\%$ of the data points.

4. Use a calculator to determine the

- mean,
- variance,
- and standard deviation

of the following data sets:

a) $8 ; 3 ; 10 ; 7 ; 7 ; 1 ; 3 ; 1 ; 3 ; 7$

Solution:

- Mean = 5
- $\sigma^2 = 9$
- $\sigma = \pm 3$

b) $4 ; 4 ; 13 ; 9 ; 7 ; 7 ; 2 ; 5 ; 15 ; 4 ; 22 ; 11$

Solution:

- Mean = $8{,}58$
- $\sigma^2 = 30{,}91$
- $\sigma = \pm 5{,}56$

c) $4{,}38 ; 3{,}83 ; 4{,}99 ; 4{,}05 ; 2{,}88 ; 4{,}83 ; 0{,}88 ; 5{,}33 ; 3{,}49 ; 4{,}10$

Solution:

- Mean = $3{,}88$
- $\sigma^2 = 1{,}47$
- $\sigma = \pm 1{,}21$

d) $4{,}76 ; -4{,}96 ; -6{,}35 ; -3{,}57 ; 0{,}59 ; -2{,}18 ; -4{,}96 ; -3{,}57 ; -2{,}18 ; 1{,}98$

Solution:

- Mean = $-1{,}66$
- $\sigma^2 = 11{,}47$
- $\sigma = \pm 3{,}39$

e) $7 ; 53 ; 29 ; 42 ; 12 ; 111 ; 122 ; 79 ; 83 ; 5 ; 69 ; 45 ; 23 ; 77$

Solution:

- Mean $= 54{,}07$
- $\sigma^2 = 1406{,}07$
- $\sigma = \pm 37{,}50$

5. Xolani surveyed the price of a loaf of white bread at two different supermarkets. The data, in rands, are given below.

Supermarket A	3,96	3,76	4,00	3,91	3,69	3,72
Supermarket B	3,97	3,81	3,52	4,08	3,88	3,68

a) Find the mean price at each supermarket and then state which supermarket has the lower mean.

Solution:

Supermarket A: $3{,}84$. Supermarket B: $3{,}82$. Supermarket B has the lower mean.

b) Find the standard deviation of each supermarket's prices.

Solution:

Standard deviation:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}$$

For Supermarket A:

$$\begin{aligned}\sigma &= \sqrt{\frac{\sum_{i=1}^{n}(x_i - 3{,}84)^2}{6}}\\ &= \sqrt{\frac{0{,}0882}{6}}\\ &= \sqrt{0{,}0147}\\ &\approx \pm 0{,}121\end{aligned}$$

For Supermarket B:

$$\begin{aligned}\sigma &= \sqrt{\frac{\sum_{i=1}^{n}(x_i - 3{,}82)^2}{6}}\\ &= \sqrt{\frac{0{,}20\dot{3}}{6}}\\ &= \sqrt{0{,}033\dot{8}}\\ &\approx \pm 0{,}184\end{aligned}$$

c) Which supermarket has the more consistently priced white bread? Give reasons for your answer.

Solution:

The standard deviation of Supermarket A's prices is lower than that of Supermarket B's. That means that Supermarket A has more consistent (less variable) prices than Supermarket B.

6. The times for the 8 athletes who swam the 100 m freestyle final at the 2012 London Olympic Games are shown below. All times are in seconds.

$47{,}52$; $47{,}53$; $47{,}80$; $47{,}84$; $47{,}88$; $47{,}92$; $48{,}04$; $48{,}44$

a) Calculate the mean time.

Solution: $\overline{x} = 47{,}87$

b) Calculate the standard deviation for the data.

Solution: $\sigma = \pm 0{,}27$

c) How many of the athletes' times are more than one standard deviation away from the mean?

Solution:

The mean is $47{,}87$ and the standard deviation is $0{,}56$. Therefore the interval containing all values that are one standard deviation from the mean is $[47{,}87 - 0{,}27; 47{,}87 + 0{,}27] = [47{,}60; 48{,}15]$. We are asked how many values are **further** than one standard deviation from the mean, meaning **outside** the interval. There are 3 values from the data set outside the interval.

7. The following data set has a mean of $14{,}7$ and a variance of $10{,}01$.

$$18\ ;\ 11\ ;\ 12\ ;\ a\ ;\ 16\ ;\ 11\ ;\ 19\ ;\ 14\ ;\ b\ ;\ 13$$

Calculate the values of a and b.

Solution:

From the formula of the mean we have

$$\begin{aligned} 14{,}7 &= \frac{114 + a + b}{10} \\ \therefore a + b &= 147 - 114 \\ \therefore a &= 33 - b \end{aligned}$$

From the formula of the variance we have

$$\begin{aligned} \sigma^2 &= \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n} \\ \therefore 10{,}01 &= \frac{69{,}12 + (a - 14{,}7)^2 + (b - 14{,}7)^2}{10} \end{aligned}$$

Substitute $a = 33 - b$ into this equation to get

$$\begin{aligned} 10{,}01 &= \frac{69{,}12 + (18{,}3 - b)^2 + (b - 14{,}7)^2}{10} \\ \therefore 100{,}1 &= 2b^2 - 66b + 620{,}1 \\ \therefore 0 &= b^2 - 33b + 260 \\ &= (b - 13)(b - 20) \end{aligned}$$

Therefore $b = 13$ or $b = 20$.

Since $a = 33 - b$ we have $a = 20$ or $a = 13$. So, the two unknown values in the data set are 13 and 20.

We do not know which of these is a and which is b since the mean and variance tell us nothing about the order of the data.

8. The height of each learner in a class was measured and it was found that the mean height of the class was $1{,}6$ m. At the time, three learners were absent. However, when the heights of the learners who were absent were included in the data for the class, the mean height did not change.

If the heights of two of the learners who were absent are $1{,}45$ m and $1{,}63$ m, calculate the height of the third learner who was absent. [NSC Paper 3 Feb-March 2013]

Solution:

Let the number of learners who were first measured be x.

The total measure of all heights is $1{,}6x$.

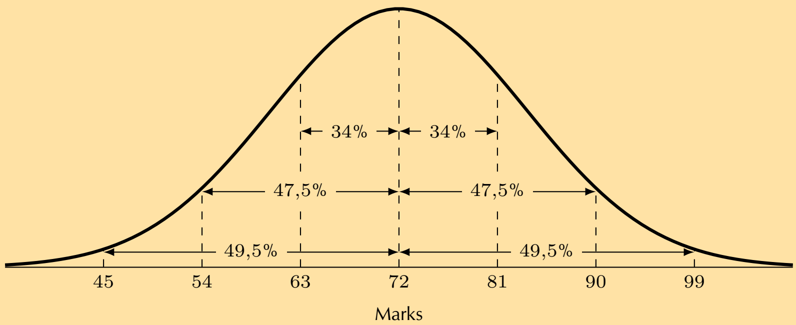
Let the height of the last learner be y.

$$\begin{aligned} \frac{1{,}6x + 1{,}45 + 1{,}63 + y}{x + 3} &= 1{,}6 \\ 1{,}6x + 3{,}08 + y &= 1{,}6x + 4{,}8 \\ y &= 1{,}72 \end{aligned}$$

9. There are 184 students taking Mathematics in a first-year university class. The marks, out of 100, in the half-yearly examination are normally distributed with a mean of 72 and a standard deviation of 9. [NSC Paper 3 Feb-March 2013]

a) What percentage of students scored between 72 and 90 marks?

Solution:



$90 = 72 + 2(9)$
Therefore 90 lies at 2 standard deviations to the right of the mean.
Hence, $47{,}5\%$ of students scored between 72 and 90 marks.

b) Approximately how many students scored between 45 and 63 marks?

Solution:

$45 = 72 - 3(9)$
$\therefore$ 45 lies at 3 standard deviations to the left of the mean.
$63 = 72 - 9$
$\therefore$ 63 lies at 1 standard deviation to the left of the mean.
The area between 1 standard deviation and 3 standard deviations is:
$49{,}5 - 34 = 15{,}5\%$
$\therefore 15{,}5\%$ of $184 \approx 29$ students scored between 45 and 63 marks.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 29BW	1b. 29BX	1c. 29BY	1d. 29BZ	1e. 29C2	1f. 29C3
2a. 29C4	2b. 29C5	2c. 29C6	2d. 29C7	2e. 29C8	2f. 29C9
3a. 29CB	3b. 29CC	3c. 29CD	4a. 29CF	4b. 29CG	4c. 29CH
4d. 29CJ	4e. 29CK	5. 29CM	6. 29CN	7. 29CP	8. 29CQ
9. 29CR					

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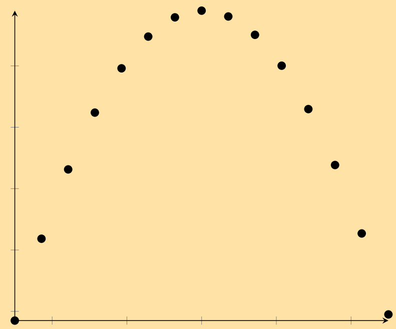
9.2 Curve fitting

Intuitive curve fitting

Exercise 9 – 2: Intuitive curve fitting

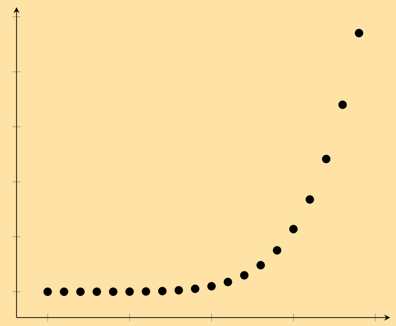
1. Identify the function (linear, exponential or quadratic) which would best fit the data in each of the scatter plots below:



a)

Solution:

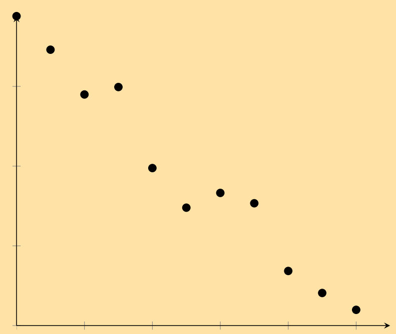
quadratic



b)

Solution:

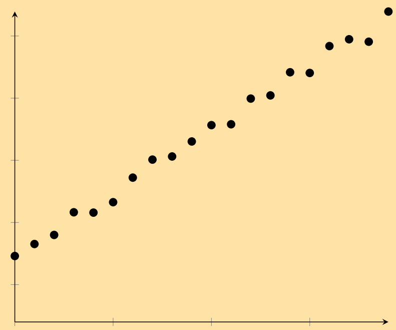
exponential



c)

Solution:

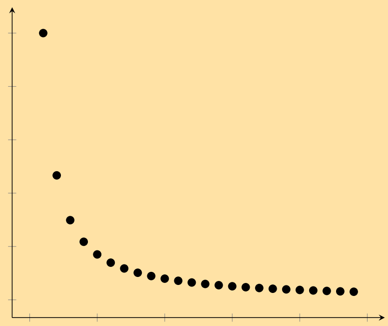
linear



d)

Solution:

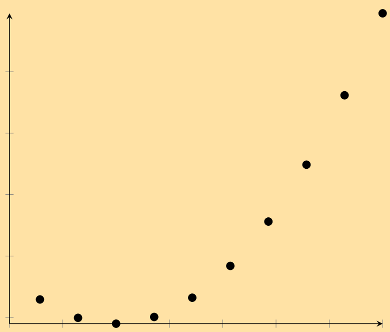
linear



e)

Solution:

exponential

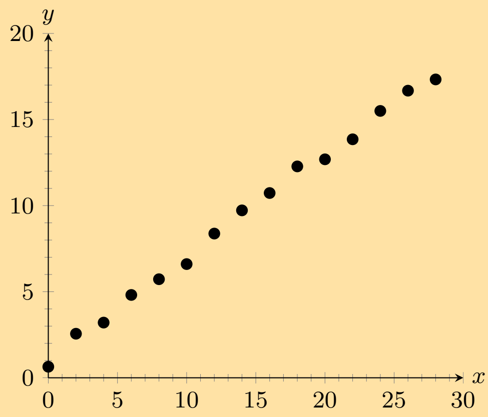


f)

Solution:

quadratic

2. Given the scatter plot below, answer the questions that follow.



a) What type of function fits the data best? Comment on the fit of the function in terms of strength and direction.

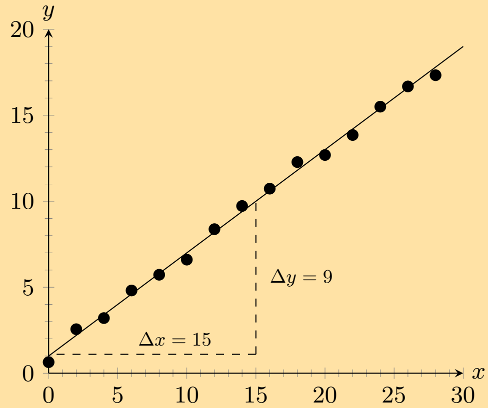
Solution:

The data fit a strong, positive linear function.

b) Draw a line of best fit through the data and determine the equation for your line.

Solution:

NB: The answer to this question is learner dependent. The method is more important than the final answer. Pay special attention to the y-intercept of the line of best fit. Learners often draw their line through the origin, even when this is not appropriate. Below is an illustration of how the learner should go about finding the solution to this problem. The learner's answer does not have to look exactly like the model answer, but should at least be a good approximation.



The y-intercept is approximately 1. The y-value at $x = 15$ is approximately 10. Therefore, $m = \frac{\Delta y}{\Delta x} = \frac{10-1}{15-0} = 0{,}6$

The equation for the line of best fit: $y = 0{,}6x + 1$

c) Using your equation, determine the estimated y-value where $x = 25$.

Solution:

Answer will depend on the learner's previous answer.

$$\begin{aligned} y &= 0{,}6(25) + 1 \\ \therefore y &= 16 \end{aligned}$$

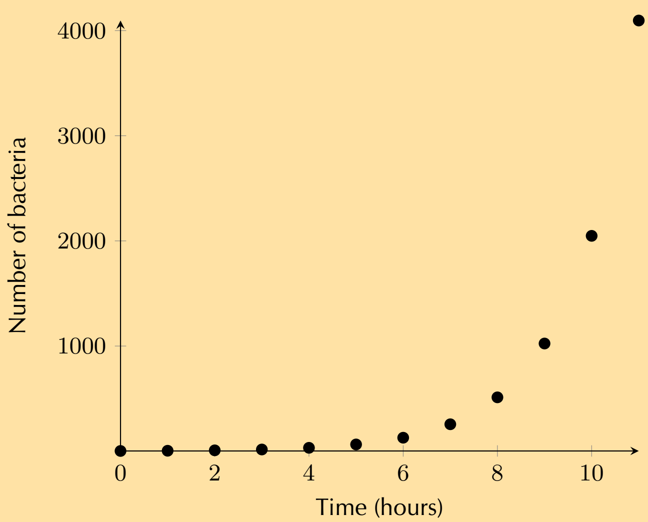
d) Using your equation, determine the estimated x-value where $y = 25$.

Solution:

Answer will depend on the learner's previous answer.

$$\begin{aligned} 25 &= 0{,}6x + 1 \\ \therefore 0{,}6x &= 24 \\ \therefore x &= \frac{24}{0{,}6} = 40 \end{aligned}$$

3. Tuberculosis (TB) is a disease of the lungs caused by bacteria which are spread through the air when an infected person coughs or sneezes. Drug-resistant TB arises when patients do not take their medication properly. Andile is a scientist studying a new treatment for drug-resistant TB. For his research, he needs to grow the TB bacterium. He takes two bacteria and puts them on a plate with nutrients for their growth. He monitors how the number of bacteria increases over time. Look at his data in the scatter plot below and answer the questions that follow.



a) What type of function do you think fits the data best?

Solution:

Exponential

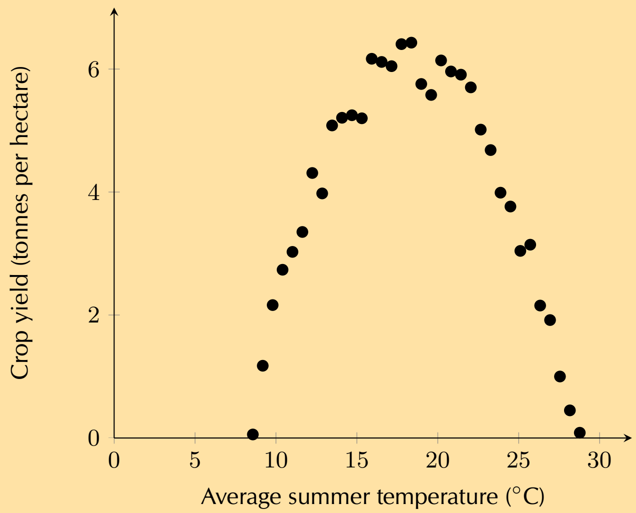
b) The equation for bacterial growth is $x_n = x_0(1+r)^t$ where x_0 is the initial number of bacteria, r is the growth rate per unit time as a proportion of 1, t is time in hours, and x_n is the number of bacteria at time, t. Determine the number of bacteria grown by Andile after 24 hours if the number of bacteria doubles every hour (i.e. the growth rate is 100% per hour).

Solution:

We are told that $x_0 = 2, t = 24$ and $r = 1$:

$$\begin{aligned} x_{24} &= 2 \times 2^{24} \\ &= 33\,554\,432 \end{aligned}$$

4. Marelize is a researcher at the Department of Agriculture. She has noticed that farmers across the country have very different crop yields depending on the region. She thinks that this has to do with the different climate in each region. In order to test her idea, she collected data on crop yield and average summer temperatures from a number of farmers. Examine her data below and answer the questions that follow.



a) Identify what type of function would fit the data best.

Solution:

Quadratic

b) Marelize determines that the equation for the function which fits the data best is $y = -0{,}06x^2 + 2{,}2x - 14$. Determine the optimal temperature to grow wheat and the respective crop yield. Round your answer to two decimal places.

Solution:

This question requires us to find the turning point of the function. There are a number of ways to do this; two are shown below:

The first method is using the formula $x = \frac{-b}{2a}$:

- The first step is to write the equation in the form: $y = ax^2 + bx + c$. Our equation is already in this form, so we can immediately substitute the values into the formula for x.

$$x = \frac{-b}{2a} = \frac{-2{,}2}{(2 \times -0{,}06)} = 18{,}33$$

- To find y, we substitute our x-value into the quadratic equation:

$$y = -0{,}06(18{,}33^2) + 2{,}2(18{,}33) - 14 = 6{,}17$$

Another method is using differentiation:

- The first step is to write the equation in the form: $y = ax^2 + bx + c$. Our equation is already in this form, so we can immediately differentiate the equation.

$$y' = -0{,}06(2)x + 2{,}2 = -0{,}12x + 2{,}2$$

- At the turning point, $y' = 0$, therefore we can now solve for x:

$$0 = -0{,}12x + 2{,}2$$
$$\therefore x = \frac{-2{,}2}{-0{,}12} = 18{,}33$$

- The x-value can now be substituted into the quadratic equation to find y:

$$y = -0{,}06(18{,}33)^2 + 2{,}2(18{,}33) - 14 = 6{,}17$$

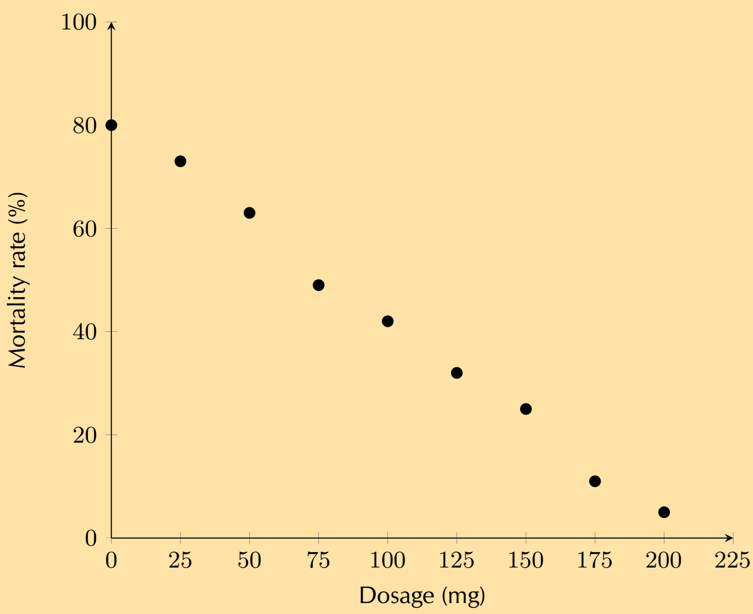
Therefore the optimal temperature to grow wheat is $18{,}33$°C and the respective crop yield is $6{,}17$ tonnes per hectare.

5. Dr Dandara is a scientist trying to find a cure for a disease which has an 80% mortality rate, i.e. 80% of people who get the disease will die. He knows of a plant which is used in traditional medicine to treat the disease. He extracts the active ingredient from the plant and tests different dosages (measured in milligrams) on different groups of patients. Examine his data below and complete the questions that follow.

Dosage (mg)	0	25	50	75	100	125	150	175	200
Mortality rate (%)	80	73	63	49	42	32	25	11	5

a) Draw a scatter plot of the data

Solution:



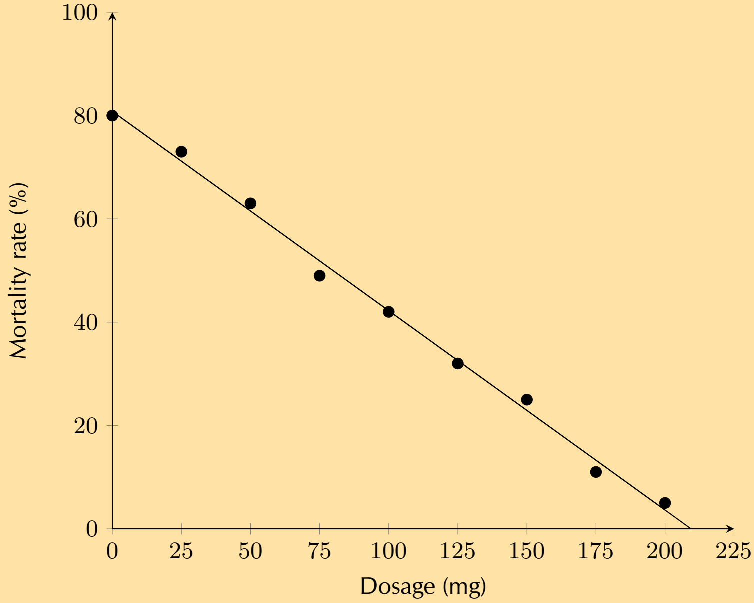
b) Which function would best fit the data? Describe the fit in terms of strength and direction.

Solution:

The data show a strong, negative linear relationship.

c) Draw a line of best fit through the data and determine the equation of your line.

Solution:



The y-intercept is approximately 80. The x-intercept is approximately 210. Therefore, $m = \frac{\Delta y}{\Delta x} = \frac{80-0}{0-210} = -0{,}38$

The equation for the line of best fit: $y = -0{,}38x + 80$

d) Use your equation to estimate the dosage required for a 0% mortality rate.

Solution:

$$0 = -0{,}38x + 80$$
$$\therefore x = \frac{-80}{-0{,}38} = 210{,}53 \text{ mg}$$

e) Dr Dandara decided to administer the estimated dosage required for a 0% mortality rate to a group of infected patients. However, he still found a mortality rate of 5%. Name the statistical technique Dr Dandara used to estimate a mortality rate of 0% and explain why his equation did not accurately predict his experimental results.

Solution:

Dr Dandara used **extrapolation** to calculate the dosage where the mortality rate $= 0\%$. Extrapolation can result in incorrect estimates if the trend observed within the available data range does not continue outside of the range. In this case, it appears that at dosages greater than 200 mg, the equation of the line of best fit no longer fits the data, therefore extrapolation produced a false estimate.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 29CS 1b. 29CT 1c. 29CV 1d. 29CW 1e. 29CX 1f. 29CY
2. 29CZ 3. 29D2 4. 29D3 5. 29D4

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Linear regression

Exercise 9 – 3: Least squares regression analysis

1. Determine the equation of the least-squares regression line using a table for the data sets below. Round a and b to two decimal places.

a)

x	10	4	9	11	11	6	8	18	9	13
y	1	0	6	3	9	5	9	8	7	15

Solution:

x	y	xy	x^2
10	1	10	100
4	0	0	16
9	6	54	81
11	3	33	121
11	9	99	121
6	5	30	36
8	9	72	64
18	8	144	324
9	7	63	81
13	15	195	169
$\sum = 99$	$\sum = 63$	$\sum = 700$	$\sum = 1113$

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2} = \frac{10 \times 700 - 99 \times 63}{10 \times 1113 - 99^2} = 0{,}574$$

$$a = \bar{y} - b\bar{x} = \frac{63}{10} - \frac{0{,}574 \times 99}{10} = 0{,}616$$

$$\therefore \hat{y} = 0{,}62 + 0{,}57x$$

b)

x	8	12	12	7	6	14	8	14	14	17
y	-5	4	3	-3	-5	-6	-2	0	-4	3

Solution:

x	y	xy	x^2
8	-5	-40	64
12	4	48	144
12	3	36	144
7	-3	-21	49
6	-5	-30	36
14	-6	-84	196
8	-2	-16	64
14	0	0	196
14	-4	-56	196
17	3	51	289
$\sum = 112$	$\sum = -15$	$\sum = -112$	$\sum = 1378$

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2} = \frac{10 \times -112 - 112 \times -15}{10 \times -1378 - 112^2} = 0{,}453$$

$$a = \bar{y} - b\bar{x} = \frac{-15}{10} - \frac{0{,}453 \times 112}{10} = -6{,}574$$

$$\therefore \hat{y} = -6{,}57 + 0{,}45x$$

c)

x	-9	3	4	7	13	6	0	8	1	14
y	0	-12	-10	-14	-31	-32	-41	-52	-51	-63

Solution:

x	y	xy	x^2
-9	0	0	81
3	-12	-36	9
4	-10	-40	16
7	-14	-98	49
13	-31	-403	169
6	-32	-192	36
0	-41	0	0
8	-52	-416	64
1	-51	-51	1
14	-63	-882	196
$\sum = 47$	$\sum = -306$	$\sum = -2118$	$\sum = 621$

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2} = \frac{10 \times -2118 - 47 \times -306}{10 \times 621 - 47^2} = -1{,}699$$

$$a = \bar{y} - b\bar{x} = \frac{-306}{10} + \frac{1{,}699 \times 47}{10} = -22{,}6147$$

$$\therefore \hat{y} = -22{,}61 - 1{,}70x$$

2. Use your calculator to determine the equation of the least squares regression line for the following sets of data:

a)

x	0,16	0,32	3	2,6	6,12	7,68	6,16	8,56	11,24	11,96
y	5,48	10,56	13,4	15,96	15,44	16,6	17,2	22,28	22,04	24,32

Solution:

$\hat{y} = 9{,}07 + 1{,}26x$

b)

x	−3,5	5,5	4	1	5,5	5	3,5	5,5	7,5	8,5
y	−10	−20,5	−30,5	−46	−46,5	−64,5	−67	−76,5	−83,5	−94

Solution:

$\hat{y} = -29{,}09 + -5{,}84x$

c)

x	2,5	4,5	−2	9	8,5	10	7,5	3	8	15
y	−2	6	11	11,5	17	21	21	30,5	32,5	33,5

Solution:

$\hat{y} = 9{,}45 + 1{,}33x$

d)

x	7,24	8,24	5,34	1,66	0,32	11,46	9,34	14,24	12,9	12,34
y	−3,2	−18,78	−21,1	−32	−31,2	−53,02	−53	−65,46	−74,8	−80,24

Solution:

$\hat{y} = -12{,}44 + -3{,}71x$

e)

x	−0,28	2,32	0,12	4,64	3,08	7,92	5,08	8,96	10,28	7,12
y	−6,88	−0,32	3,68	4,8	11,68	19,2	20,96	24,96	29,28	33,28

Solution:

$\hat{y} = -1{,}94 + 3{,}25x$

f)

x	1	1,1	4,8	3,55	2,75	1,95	6,1	8,9	10,35	9,55
y	−8,45	−5,95	−4,35	0,85	−2,95	−1,8	0,25	0,05	4,8	−3,05

Solution:

$\hat{y} = -5{,}64 + 0{,}72x$

g)

x	1,9	1,1	−1,5	1,3	0,95	8,25	10,6	6,2	8,1	8,65
y	7	8,45	0,9	0,1	2,45	4,35	2,2	1,4	0,15	2,05

Solution:

$\hat{y} = 3{,}52 + -0{,}13x$

h)

x	−81,8	73,1	84	92,2	−69,7	−56,1	8,8	80,9	68,4	−40,4
y	10,6	16,1	3,6	4,6	11,9	18,3	16,6	17,6	17,7	24,1

Solution:

$\hat{y} = 14{,}55 + -0{,}03x$

i)

x	2,8	7,4	−2,4	4	11,3	6,9	2,5	1,7	5,4	8,2
y	12,4	13,4	15,3	15,4	16,4	19,2	21,1	19,4	21,3	25

Solution:

$\hat{y} = 16{,}94 + 0{,}20x$

j)

x	5	1,2	8	6	7,4	7,4	6,7	8,7	12,2	14,3
y	−4,2	−13,7	−23,7	−33,5	−43,8	−54,2	−63,9	−73,9	−84,5	−93,5

Solution:

$\hat{y} = 5{,}14 + -7{,}03x$

3. Determine the equation of the least squares regression line given each set of data values below. Round a and b to two decimal places in your final answer.

a) $n = 10$; $\sum x = 74$; $\sum y = 424$; $\sum xy = 4114{,}51$; $\sum(x^2) = 718{,}86$

Solution:

$$b = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n}(x_i)^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$= \frac{10 \times 4114{,}51 - 74 \times 424}{10 \times 718{,}86 - 74^2} = 5{,}704250847$$

$$a = \bar{y} - b\bar{x} = \frac{424}{10} - 5{,}704250847 \times \frac{74}{10} = 0{,}188543732$$

$$\therefore \hat{y} = 0{,}19 + 5{,}70x$$

b) $n = 13;\ \ \bar{x} = 8{,}45;\ \ \bar{y} = 17{,}83;\ \ \sum xy = 1879{,}25;\ \ \sum(x^2) = 855{,}45$

Solution:

$$\bar{x} = \frac{\sum_{i=1}^{n}(x_i)}{n}$$

$$\therefore \bar{x}n = \sum_{i=1}^{n}(x_i)$$

$$\therefore b = \frac{n\sum_{i=1}^{n} x_i y_i - (\bar{x}n)(\bar{y}n)}{\sum_{i=1}^{n} y_i n \sum_{i=1}^{n}(x_i)^2 - (\bar{x}n)^2}$$

$$= \frac{13 \times 1879{,}25 - (13 \times 8{,}45) \times (13 \times 17{,}83)}{13 \times 855{,}45 - (13 \times 8{,}45)^2} = 1{,}090584962$$

$$a = \bar{y} - b\bar{x} = 17{,}83 - 1{,}090584962 \times 8{,}45 = 8{,}614557071$$

$$\therefore \hat{y} = 8{,}61 + 1{,}09x$$

c) $n = 10;\ \ \bar{x} = 5{,}77;\ \ \bar{y} = 17{,}03;\ \ \overline{xy} = 133{,}817;\ \ \sigma_x = \pm 3{,}91$
(Hint: multiply the numerator and denominator of the formula for b by $\frac{1}{n^2}$)

Solution:

$$Var[x] = \sigma_x^2 = \frac{\sum_{i=1}^{n} x_i^2}{n} - \bar{x}^2 \text{ (proof below the solution)}$$

$$\bar{x} = \frac{\sum_{i=1}^{n}(x_i)}{n}$$

$$\therefore b \times \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{\frac{\sum_{i=1}^{n} x_i y_i}{n} - \frac{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n^2}}{\frac{\sum_{i=1}^{n} {x_i}^2}{n} - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n^2}} = \frac{\overline{xy} - \bar{x}\bar{y}}{\frac{\sum_{i=1}^{n} {x_i}^2}{n} - \bar{x}^2} = \frac{\overline{xy} - \bar{x}\bar{y}}{Var[x]}$$

$$= \frac{133{,}817 - (5{,}77 \times 17{,}03)}{3{,}91^2} = 2{,}325593108$$

$$a = \bar{y} - b\bar{x} = 17{,}03 - 2{,}325593108 \times 5{,}77 = 3{,}611327767$$

$$\therefore \hat{y} = 3{,}61 + 2{,}33x$$

$$\text{RTP: } Var[x] = \frac{\sum_{i=1}^{n} x_i^2}{n} - \bar{x}^2$$

$$Var[x] = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n} \text{ (from the formula)}$$

$$= \frac{\sum_{i=1}^{n} (x_i^2 - 2x_i\bar{x} - \bar{x}^2)}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - 2\bar{x}\frac{\sum_{i=1}^{n} x_i}{n} + \frac{\sum_{i=1}^{n} \bar{x}^2}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - 2\bar{x}^2 + \frac{\cancel{n}\bar{x}^2}{\cancel{n}}$$

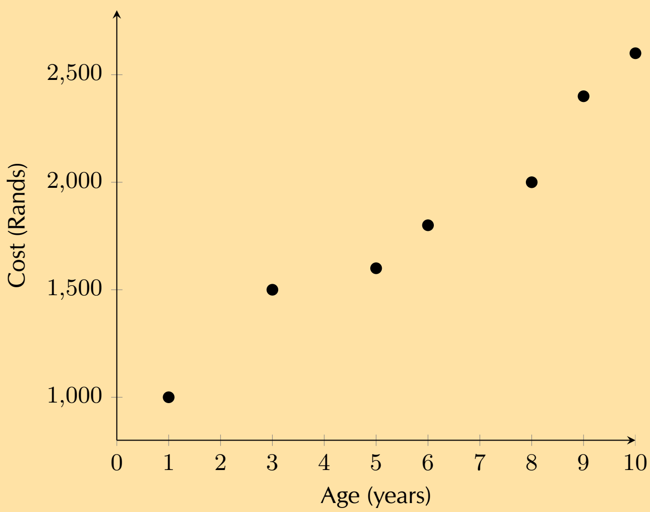
$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - \bar{x}^2$$

4. The table below shows the average maintenance cost in rands of a certain model of car compared to the age of the car in years.

Age (x)	1	3	5	6	8	9	10
Cost (y)	1000	1500	1600	1800	2000	2400	2600

a) Draw a scatter plot of the data.

Solution:



b) Complete the table below, filling in the totals of each column in the final row:

Age (x)	**Cost** (y)	xy	x^2
1	1000		
3	1500		
5	1600		
6	1800		
8	2000		
9	2400		
10	2600		
$\sum = \ldots$	$\sum = \ldots$	$\sum = \ldots$	$\sum = \ldots$

Solution:

Age (x)	**Cost** (y)	xy	x^2
1	1000	1000	1
3	1500	4500	9
5	1600	8000	25
6	1800	10 800	36
8	2000	16 000	64
9	2400	21 600	81
10	2600	26 000	100
$\sum = 42$	$\sum = 12\,900$	$\sum = 87\,900$	$\sum = 316$

c) Use your table to determine the equation of the least squares regression line. Round a and b to two decimal places.

Solution:

$$b = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n}(x_i)^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$
$$= \frac{7 \times 87\ 900 - 42 \times 12\ 900}{7 \times 326 - 42^2} = 164{,}0625$$

$$a = \bar{y} - b\bar{x} = \frac{12\ 900}{7} - 164{,}0625 \times \frac{42}{7} = 858{,}48$$

$$\therefore \hat{y} = 858{,}48 + 164{,}06x$$

d) Use your equation to estimate what it would cost to maintain this model of car in its 15^{e} year.

Solution:

$$y = 858{,}48 + 164{,}06(15) = \text{R}\ 3319{,}42$$

e) Use your equation to estimate the age of the car in the year where the maintenance cost totals over R 3000 for the first time.

Solution:

$$3000 = 858{,}48 + 164{,}06x$$
$$2141{,}52 = 164{,}06x$$
$$x = \frac{2141{,}52}{164{,}06} = 13{,}05$$

Therefore the maintenance cost will exceed R 3000 for the first time when the car is aged 13.

5. Miss Colly has always maintained that there is a relationship between a learner's ability to understand the language of instruction and their marks in Mathematics. Since she teaches Mathematics through the medium of English, she decides to compare the Mathematics and English marks of her learners in order to investigate the relationship between the two marks. A sample of her data is shown in the table below:

English % (x)	28	33	30	45	45	55	55	65	70	76	65	85	90
Mathematics % (y)	35	36	34	45	50	40	60	50	65	85	70	80	90

a) Complete the table below, filling in the totals of each column in the final row:

English % (x)	Mathematics % (y)	xy	x^2
28	35		
33	36		
30	34		
45	45		
45	50		
55	40		
65	50		
70	65		
76	85		
65	70		
85	80		
90	90		
$\sum = \ldots$	$\sum = \ldots$	$\sum = \ldots$	$\sum = \ldots$

Solution:

English % (x)	Mathematics % (y)	xy	x^2
28	35	980	784
33	36	1188	1089
30	34	1020	900
45	45	2025	2025
45	50	2250	2025
55	40	2200	3025
65	50	3250	4225
70	65	4550	4900
76	85	6460	5776
65	70	4550	4225
85	80	6800	7225
90	90	8100	8100
$\sum = 742$	$\sum = 740$	$\sum = 46\ 673$	$\sum = 47\ 324$

b) Use your table to determine the equation of the least squares regression line. Round a and b to two decimal places.

Solution:

$$b = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n}(x_i)^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$
$$= \frac{13 \times 46\ 673 - 742 \times 740}{13 \times 47\ 324 - 742^2} = 0{,}8920461577$$

$$a = \bar{y} - b\bar{x} = \frac{740}{13} - 0{,}8920461577 \times \frac{742}{13} = 6{,}007827002$$

$$\therefore \hat{y} = 6{,}01 + 0{,}89x$$

c) Use your equation to estimate the Mathematics mark of a learner who obtained 50% for English, correct to two decimal places.

Solution:

$$y = 6{,}01 + 0{,}89(50) = 50{,}51\%$$

d) Use your equation to estimate the English mark of a learner who obtained 75% for Mathematics, correct to two decimal places.

Solution:

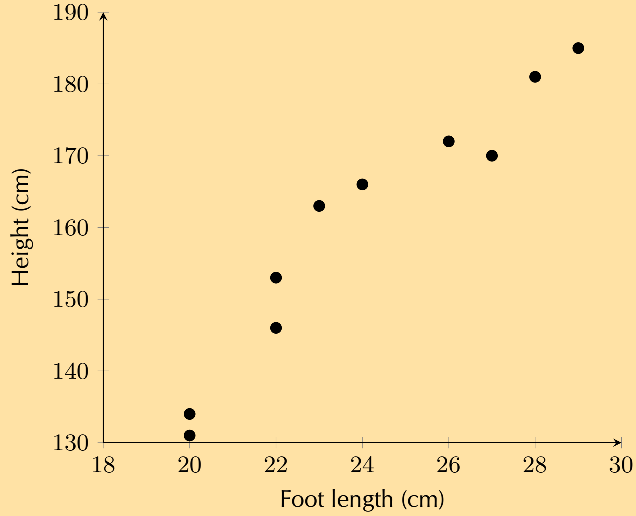
$$75 = 6{,}01 + 0{,}89x$$
$$68{,}99 = 0{,}89x$$
$$x = \frac{68{,}99}{0{,}89} = 77{,}52\%$$

6. Foot lengths and heights of ten students are given in the table below.

Height (cm)	170	163	131	181	146	134	166	172	185	153
Foot length (cm)	27	23	20	28	22	20	24	26	29	22

a) Using foot length as your x-variable, draw a scatter plot of the data.

Solution:



b) Identify and describe any trends shown in the scatter plot.

Solution:

Strong (or fairly strong), positive, linear trend

c) Find the equation of the least squares line using the formulae and draw the line on your graph. Round a and b to two decimal places in your final answer.

Solution:

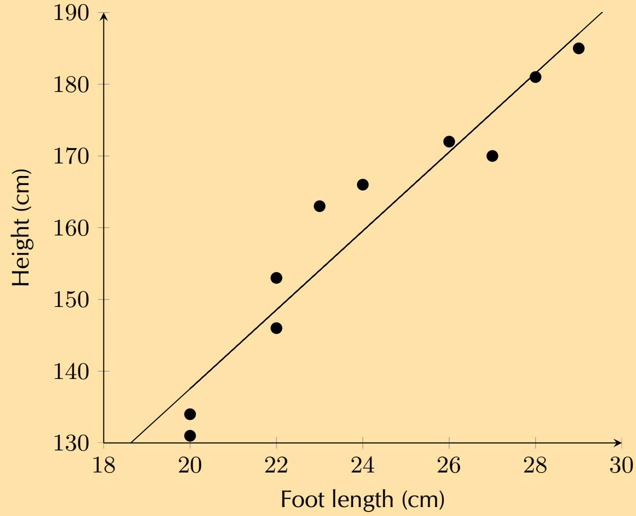
Foot length (x)	**Height** (y)	xy	x^2
27	170	4590	729
23	163	3749	529
20	131	2620	400
28	181	5068	784
22	146	3212	484
20	134	2680	400
24	166	3984	576
26	172	4472	676
29	185	5365	841
22	153	3366	484
$\sum = 241$	$\sum = 1601$	$\sum = 39\ 106$	$\sum = 5903$

$$b = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n}(x_i)^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$= \frac{10 \times 39\ 106 - 241 \times 1601}{10 \times 5903 - 241^2} = 5{,}49947313$$

$$a = \bar{y} - b\bar{x} = \frac{1601}{10} - 5{,}49947313 \times \frac{241}{10} = 27{,}56269575$$

$$\therefore \hat{y} = 27{,}56 + 5{,}50x$$



d) Confirm your calculations above by finding the least squares regression line using a calculator.

Solution: Answer should be the same as c).

e) Use your equation to predict the height of a student with a foot length of 21,6 cm.

Solution:

$$y = 27{,}56 + 5{,}5(21{,}6) = 146{,}36 \text{ cm}$$

f) Use your equation to predict the foot length of a student 190 cm tall, correct to two decimal places.

Solution:

$$190 = 5{,}5x + 27{,}56$$
$$\therefore x = \frac{162{,}44}{5{,}5} = 29{,}53 \text{ cm}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 29D6	1b. 29D7	1c. 29D8	2a. 29D9	2b. 29DB	2c. 29DC
2d. 29DD	2e. 29DF	2f. 29DG	2g. 29DH	2h. 29DJ	2i. 29DK
2j. 29DM	3a. 29DN	3b. 29DP	3c. 29DQ	4. 29DR	5. 29DS
6. 29DT					

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9.3 Correlation

NB. See the fifth bullet at the beginning of the chapter regarding the formula for the correlation coefficient.

Exercise 9 – 4: Correlation coefficient

1. Determine the correlation coefficient by hand for the following data sets and comment on the strength and direction of the correlation. Round your answers to two decimal places.

a)

x	5	8	13	10	14	15	17	12	18	13
y	5	8	3	8	7	5	3	-1	4	-1

Solution:

x	y	xy	x^2	$x - \bar{x}^2$	$y - \bar{y}^2$
5	5	25	25	56,25	0,81
8	8	64	64	20,25	15,21
13	3	39	169	0,25	1,21
10	8	80	100	6,25	15,21
14	7	98	196	2,25	8,41
15	5	75	225	6,25	0,81
17	3	51	289	20,25	1,21
12	−1	−12	144	0,25	26,01
18	4	72	324	30,25	0,01
13	−1	−13	169	0,25	26,01
$\sum =$ 125	$\sum =$ 41	$\sum =$ 479	$\sum =$ 1705	$\sum =$ 142,5	$\sum =$ 94,9

$$r = b\frac{\sigma_x}{\sigma_y}$$

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2} = \frac{10 \times 479 - 125 \times 41}{10 \times 1705 - 125^2} = -0{,}235$$

$$\sigma_x = \sqrt{\frac{\sum (x - \bar{x})^2}{n}} = \sqrt{\frac{142{,}5}{10}} = \sqrt{14{,}25} = \pm 3{,}775$$

$$\sigma_y = \sqrt{\frac{\sum (y - \bar{y})^2}{n}} = \sqrt{\frac{94{,}9}{10}} = \sqrt{9{,}49} = \pm 3{,}081$$

$$\therefore r = -0{,}235 \times \frac{3{,}775}{3{,}081}$$

$$= -0{,}29$$

Therefore, the correlation between x and y is negative but weak.

b)

x	7	3	11	7	7	6	9	12	10	15
y	13	23	32	45	50	55	67	69	85	90

Solution:

x	y	xy	x^2	$x - \bar{x}^2$	$y - \bar{y}^2$
7	13	91	49	2,89	1592,01
3	23	69	9	32,49	894,01
11	32	352	121	5,29	436,81
7	45	315	49	2,89	62,41
7	50	350	49	2,89	8,41
6	55	330	36	7,29	4,41
9	67	603	81	0,09	198,81
12	69	828	144	10,89	259,21
10	85	850	100	1,69	1030,41
15	90	1350	225	39,69	1376,41
$\sum =$ 87	$\sum =$ 529	$\sum =$ 5138	$\sum =$ 863	$\sum =$ 106,1	$\sum =$ 5862,9

$$r = b\frac{\sigma_x}{\sigma_y}$$

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2} = \frac{10 \times 5138 - 87 \times 529}{10 \times 863 - 87^2} = 5{,}049$$

$$\sigma_x = \sqrt{\frac{\sum (x - \bar{x})^2}{n}} = \sqrt{\frac{106{,}1}{10}} = \sqrt{10{,}61} = \pm 3{,}26$$

$$\sigma_y = \sqrt{\frac{\sum (y - \bar{y})^2}{n}} = \sqrt{\frac{5862{,}9}{10}} = \sqrt{586{,}29} = \pm 24{,}21$$

$$\therefore r = 5{,}049 \times \frac{3{,}26}{24{,}21}$$

$$= 0{,}68$$

Therefore, the correlation between x and y is positive and moderate.

c)

x	3	10	7	6	11	16	17	15	17	20
y	6	24	30	38	53	56	65	75	91	103

Solution:

x	y	xy	x^2	$x - \bar{x}^2$	$y - \bar{y}^2$
3	6	18	9	84,64	2313,61
10	24	240	100	4,84	906,01
7	30	210	49	27,04	580,81
6	38	228	36	38,44	259,21
11	53	583	121	1,44	1,21
16	56	896	256	14,44	3,61
17	65	1105	289	23,04	118,81
15	75	1125	225	7,84	436,81
17	91	1547	289	23,04	1361,61
20	103	2060	400	60,84	2391,21
$\sum =$ 122	$\sum =$ 541	$\sum =$ 8012	$\sum =$ 1774	$\sum =$ 285,6	$\sum =$ 8372,9

$$
\begin{aligned}
r &= b\frac{\sigma_x}{\sigma_y} \\
b &= \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2} = \frac{10 \times 8012 - 122 \times 541}{10 \times 1774 - 122^2} = 4{,}943 \\
\sigma_x &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} = \sqrt{\frac{285{,}6}{10}} = \sqrt{28{,}56} = \pm 5{,}344 \\
\sigma_y &= \sqrt{\frac{\sum (y - \bar{y})^2}{n}} = \sqrt{\frac{8372{,}9}{10}} = \sqrt{837{,}29} = \pm 28{,}936 \\
\therefore r &= 4{,}943 \times \frac{5{,}344}{28{,}936} \\
&= 0{,}91
\end{aligned}
$$

Therefore, the correlation between x and y is positive and very strong.

2. Using your calculator, determine the value of the correlation coefficient to two decimal places for the following data sets and describe the strength and direction of the correlation.

a)

x	0,1	0,8	1,2	3,4	6,5	3,9	6,4	7,4	9,9	8,5
y	−5,1	−10	−17,3	−24,9	−31,9	−38,6	−42	−55	−62	−64,8

Solution:

$r = -0{,}95$, negative, very strong.

b)

x	−26	−34	−51	−14	50	−57	−11	−10	36	−35
y	−66	−10	−26	−51	−58	−56	45	−142	−149	−30

Solution:

$r = -0{,}40$, negative, weak.

c)

x	101	−398	103	204	105	606	807	−992	609	−790
y	−300	98	−704	−906	−8	690	−12	686	984	−18

Solution:

$r = 0{,}00$, no correlation

d)

x	101	82	−7	−6	45	−94	−23	78	−11	0
y	111	−74	21	106	51	26	21	86	−29	66

Solution:

$r = 0{,}14$, positive, very weak.

e)

x	−3	5	−4	0	−2	9	10	11	17	9
y	24	18	21	30	31	39	48	59	56	54

Solution:

$r = 0{,}83$, positive, strong.

3. Calculate and describe the direction and strength of r for each of the sets of data values below. Round all r-values to two decimal places.

a) $b = -1{,}88$; $\sigma_x^2 = 48{,}62$; $\sigma_y^2 = 736{,}54$.

Solution:

$$r = -1{,}88 \times \sqrt{\frac{48{,}62}{736{,}54}} = -0{,}48$$

b) $a = 32{,}19$; $x = 4{,}3$; $\bar{y} = 36{,}6$; $\sum_{i=1}^{n}(x_i - \bar{x})^2 = 620{,}1$; $\sum_{i=1}^{n}(y_i - \bar{y})^2 = 2636{,}4$.

Solution:

$$a = \bar{y} - b\bar{x}$$
$$\therefore b = \frac{\hat{y} - a}{x} = \frac{36{,}6 - 32{,}19}{4{,}3} = 1{,}03$$
$$\therefore r = 1{,}03 \times \sqrt{\frac{620{,}1}{2636{,}4}} = 0{,}50$$

4. The geography teacher, Mr Chadwick, gave the data set below to his class to illustrate the concept that average temperature depends on how far a place is from the equator (known as the latitude). There are 90 degrees between the equator and the North Pole. The equator is defined as 0 degrees. Examine the data set below and answer the questions that follow.

City	**Degrees N (x)**	**Average temp. (y)**	xy	x^2	$(x - \bar{x})^2$	$(y - \bar{y})^2$
Cairo	43	22				
Berlin	53	19				
London	40	18				
Lagos	6	32				
Jerusalem	31	23				
Madrid	40	28				
Brussels	51	18				
Istanbul	39	23				
Boston	43	23				
Montreal	45	22				
Total:						

a) Copy and complete the table.

Solution:

City	**°N (x)**	**Ave. temp. (y)**	xy	x^2	$(x - \bar{x})^2$	$(y - \bar{y})^2$
Cairo	43	22	946	1849	15,21	0,64
Berlin	53	19	1007	2809	193,21	14,44
London	40	18	720	1600	0,81	23,04
Lagos	6	32	192	36	1095,61	84,64
Jerusalem	31	23	713	961	65,61	0,04
Madrid	40	28	1120	1600	0,81	27,04
Brussels	51	18	918	2601	141,61	23,04
Istanbul	39	23	897	1521	0,01	0,04
Boston	43	23	989	1849	15,21	0,04
Montreal	45	22	990	2025	34,81	0,64
Total:	391	228	8492	16 851	1562,9	173,6

b) Using your table, determine the equation of the least squares regression line. Round a and b to two decimal places in your final answer.

Solution:

$$b = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n}(x_i)^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$
$$= \frac{10 \times 8492 - 391 \times 228}{10 \times 16\,851 - 391^2} = -0{,}2705227462$$

$$a = \bar{y} - b\bar{x} = \frac{228}{10} - -0{,}2705227462 \times \frac{391}{10} = 33{,}37743938$$

$$\therefore \hat{y} = 33{,}38 - 0{,}27x$$

c) Use your calculator to confirm your equation for the least squares regression line.

Solution:

Answer should be as above.

d) Using your table, determine the value of the correlation coefficient to two decimal places.

Solution:

$$\begin{aligned} r &= b\left(\frac{\sigma_x}{\sigma_y}\right) \\ &= -0{,}27\left(\frac{\sqrt{\frac{1562{,}9}{10}}}{\sqrt{\frac{173{,}6}{10}}}\right) \\ &= -0{,}81 \end{aligned}$$

e) What can you deduce about the relationship between how far north a city is and its average temperature?

Solution:

There is a strong, negative, linear correlation between how far north a city is (latitude) and average temperature.

f) Estimate the latitude of Paris if it has an average temperature of $25°$C

Solution:

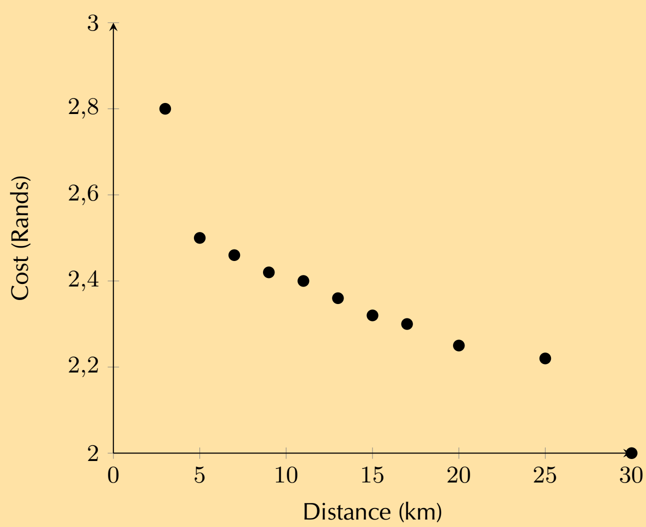
$$\begin{aligned} 25 &= 33{,}38 + -0{,}27(x) \\ \therefore x &= \frac{25 - 33{,}38}{-0{,}27} \\ &= 31{,}04 \text{ degrees North} \end{aligned}$$

5. A taxi driver recorded the number of kilometres his taxi travelled per trip and his fuel cost per kilometre in Rands. Examine the table of his data below and answer the questions that follow.

Distance (x)	3	5	7	9	11	13	15	17	20	25	30
Cost (y)	2,8	2,5	2,46	2,42	2,4	2,36	2,32	2,3	2,25	2,22	2

a) Draw a scatter plot of the data.

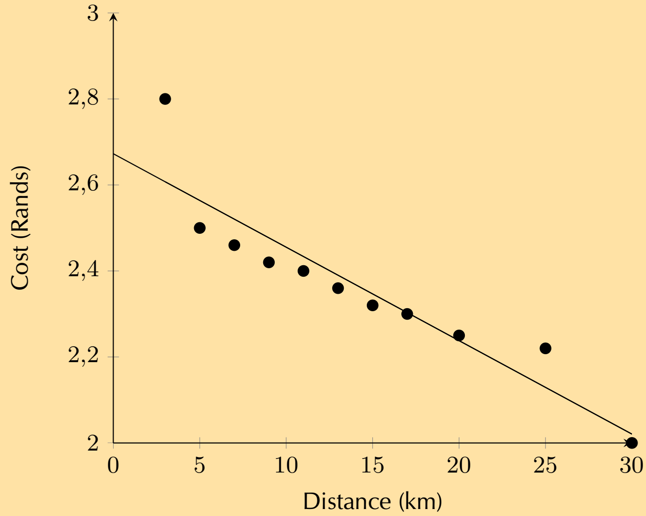
Solution:



b) Use your calculator to determine the equation of the least squares regression line and draw this line on your scatter plot. Round a and b to two decimal places in your final answer.

Solution:

$$\hat{y} = 2{,}67 + -0{,}02x$$



c) Using your calculator, determine the correlation coefficient to two decimal places.

Solution:

$r = -0{,}92$

d) Describe the relationship between the distance travelled per trip and the fuel cost per kilometre.

Solution:

There is a very strong, negative, linear relationship between distance travelled per trip and the fuel cost per kilometre.

e) Predict the distance travelled if the cost per kilometre is R $1{,}75$.

Solution:

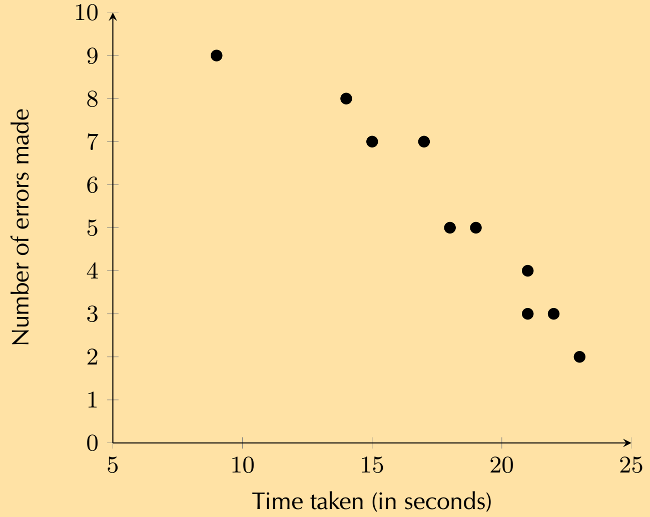
$$1{,}75 = 2{,}67 - 0{,}02x$$
$$\therefore x = \frac{1{,}75 - 2{,}67}{-0{,}02} = 46 \text{ kilometres}$$

6. The time taken, in seconds, to complete a task and the number of errors made on the task were recorded for a sample of 10 primary school learners. The data is shown in the table below. [Adapted from NSC Paper 3 Feb-March 2013]

Time taken to complete task (in seconds)	23	21	19	9	15	22	17	14	21	18
Number of errors made	2	4	5	9	7	3	7	8	3	5

a) Draw a scatter plot of the data.

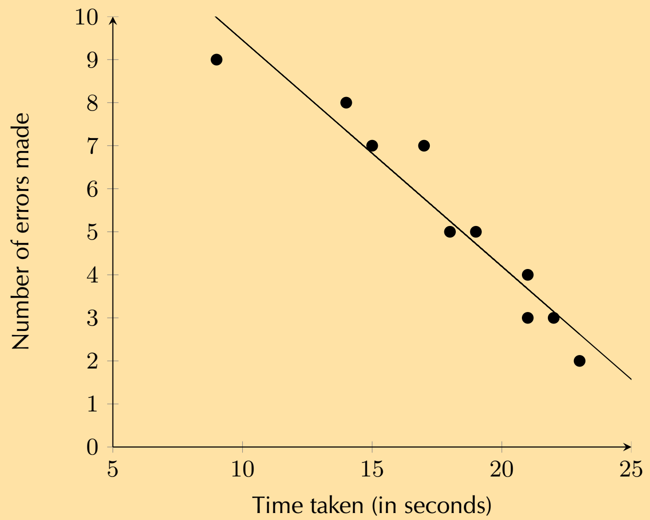
Solution:



b) What is the influence of more time taken to complete the task on the number of errors made?
Solution:
When more time is taken to complete the task, the learners make fewer errors.
OR
When less time is taken to complete the task, the learners make more errors.

c) Determine the equation of the least squares regression line and draw this line on your scatter plot. Round a and b to two decimal places in your final answer.
Solution:

$$\begin{aligned} a &= 14{,}71 \\ b &= -0{,}53 \\ \hat{y} &= 14{,}71 - 0{,}53x \end{aligned}$$



d) Determine the correlation coefficient to two decimal places.
Solution:
$r = -0{,}96$

e) Predict the number of errors that will be made by a learner who takes 13 seconds to complete this task.
Solution:

$$\begin{aligned} \hat{y} &= 14{,}71 - 0{,}53(13) \\ &\approx 7{,}82 \\ &\approx 8 \end{aligned}$$

f) Comment on the strength of the relationship between the variables.

Solution:

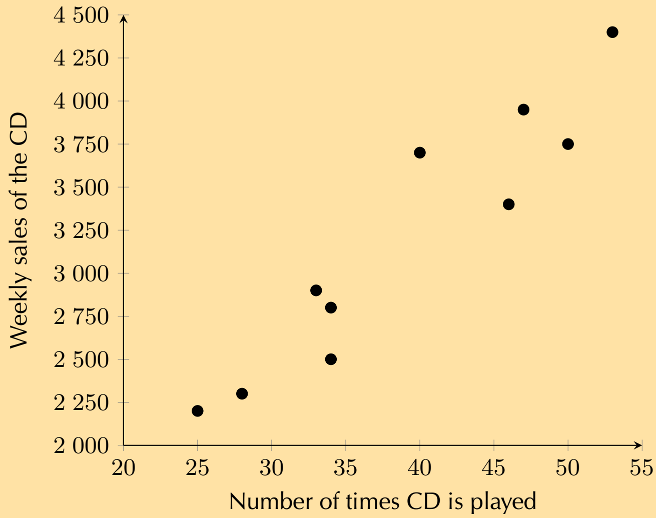
There is a strong negative relationship between the variables.

7. A recording company investigates the relationship between the number of times a CD is played by a national radio station and the national sales of the same CD in the following week. The data below was collected for a random sample of 10 CDs. The sales figures are rounded to the nearest 50. [NSC Paper 3 November 2012]

Number of times CD is played	47	34	40	34	33	50	28	53	25	46
Weekly sales of the CD	3950	2500	3700	2800	2900	3750	2300	4400	2200	3400

a) Draw a scatter plot of the data.

Solution:



b) Determine the equation of the least squares regression line.

Solution:

$$a = 293{,}06$$
$$b = 74{,}28$$
$$\hat{y} = 293{,}06 + 74{,}28x$$

c) Calculate the correlation coefficient.

Solution: $r = 0{,}95$

d) Predict, correct to the nearest 50, the weekly sales for a CD that was played 45 times by the radio station in the previous week.

Solution:

$$\begin{aligned}\hat{y} &= 293{,}06 + 74{,}28(45)\\ &= 3635{,}66\\ &\approx 3650 \text{ (to the nearest 50)}\end{aligned}$$

e) Comment on the strength of the relationship between the variables.

Solution:

There is a very strong positive relationship between the number of times that a CD was played and the sales of that CD in the following week.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 29DW 1b. 29DX 1c. 29DY 2a. 29DZ 2b. 29F2 2c. 29F3
2d. 29F4 2e. 29F5 3a. 29F6 3b. 29F7 4. 29F8 5. 29F9
6. 29FB 7. 29FC

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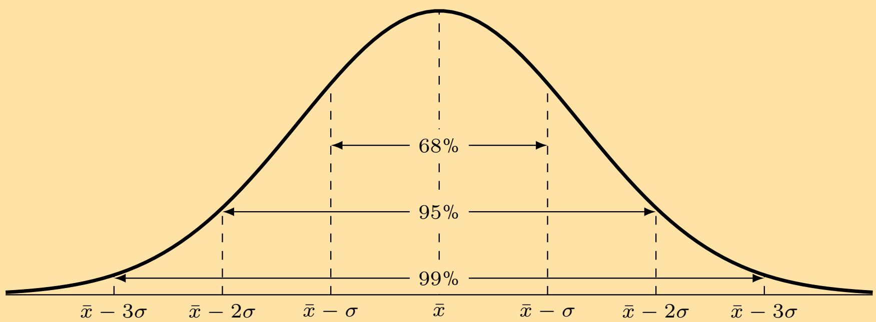
9.4 Summary

Exercise 9 – 5: End of chapter exercises

1. The number of SMS messages sent by a group of teenagers was recorded over a period of a week. The data was found to be normally distributed with a mean of 140 messages and a standard deviation of 12 messages. [NSC Paper 3 Feb-March 2012]



Answer the following questions with reference to the information provided in the graph:

a) What percentage of teenagers sent less than 128 messages?

Solution:

$140 - 12 = 128$

128 is 1 standard deviation to the left of the mean, therefore the percentage of teenagers who sent less than 128 messages is:

$50\% - 34\% = 16\%$

b) What percentage of teenagers sent between 116 and 152 messages?

Solution:

116 minutes is 2 standard deviations from the mean, therefore $47,5\%$

152 minutes is 1 standard deviation from the mean, therefore 34%

Percentage of the teenagers who sent between 116 and 152 messages $= 47,5\% + 34\% = 81,5\%$

2. A company produces sweets using a machine which runs for a few hours per day. The number of hours running the machine and the number of sweets produced are recorded.

Machine hours	Sweets produced
3,80	275
4,23	287
4,37	291
4,10	281
4,17	286

Find the linear regression equation for the data, and estimate the machine hours needed to make 300 sweets.

Solution:

Using a calculator, the equation is:

$$\hat{y} = 165{,}70 + 28{,}62x$$

Therefore, the estimated number of machine hours needed to make 300 sweets is:

$$\begin{aligned} 300 &= 165{,}70 + 28{,}62x \\ \therefore x &= \frac{300 - 165{,}7}{28{,}62} = 4{,}69 \text{ machine hours} \end{aligned}$$

3. The profits of a new shop are recorded over the first 6 months. The owner wants to predict his future sales. The profits by month so far have been R 90 000; R 93 000; R 99 500; R 102 000; R 101 300; R 109 000.

   a) Calculate the linear regression function for the data, using profit as your y-variable. Round a and b to two decimal places.

      Solution:

      $$\hat{y} = 86\ 893{,}33 + 3497{,}14x$$

   b) Give an estimate of the profits for the next two months.

      Solution:

      $$\begin{aligned} \text{Profit seventh month} &= 86\ 893{,}33 + 3497{,}14(7) = \text{R } 111\ 373{,}31 \\ \text{Profit eighth month} &= 86\ 893{,}33 + 3497{,}14(8) = \text{R } 114\ 870{,}45 \end{aligned}$$

   c) The owner wants a profit of R 130 000. Estimate how many months this will take.

      Solution:

      $$\begin{aligned} 130\ 000 &= 86\ 893{,}33 + 3497{,}14x \\ \therefore x &= \frac{130\ 000 - 86\ 893{,}33}{3497{,}14} = 12{,}33 \end{aligned}$$

      It will take 13 months to reach a profit of R 130 000.

4. A fast food company produces hamburgers. The number of hamburgers made and the costs are recorded over a week.

Hamburgers made	Costs
495	R 2382
550	R 2442
515	R 2484
500	R 2400
480	R 2370
530	R 2448
585	R 2805

   a) Find the linear regression function that best fits the data. Use hamburgers made as your x-variable and round a and b to two decimal places.

      Solution:

      $\hat{y} = 601{,}28 + 3{,}59x$

   b) Calculate the value of the correlation coefficient, correct to two decimal places, and comment on the strength and direction of the correlation.

      Solution:

      $r = 0{,}86$

      There is a strong, positive, linear correlation.

   c) If the total cost in a day is R 2500, estimate the number of hamburgers produced. Round your answer down to the nearest whole number.

Solution:

$$2500 = 601{,}28 + 3{,}59x$$
$$\therefore x = \frac{2500 - 601{,}28}{3.59} = 528{,}89$$

Therefore 528 burgers are produced.

d) What is the cost of 490 hamburgers?

Solution:

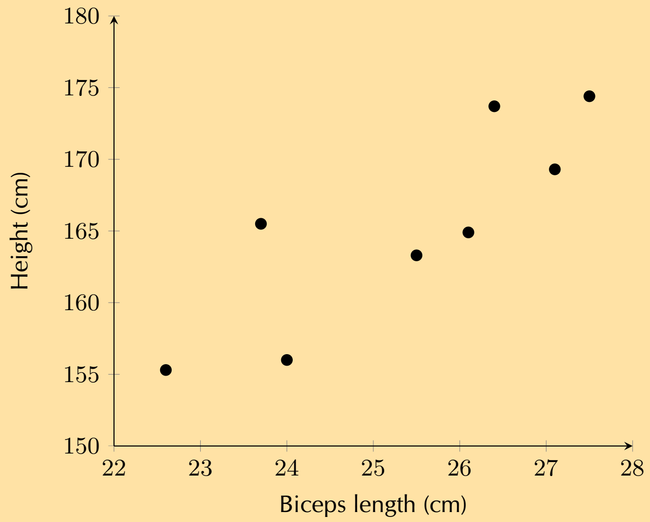
$$y = 601{,}28 + 3{,}59(490) = \text{R } 2360{,}38$$

5. A collection of data related to an investigation into biceps length and height of students was recorded in the table below. Answer the questions to follow.

Length of right biceps (cm)	Height (cm)
25,5	163,3
26,1	164,9
23,7	165,5
26,4	173,7
27,5	174,4
24	156
22,6	155,3
27,1	169,3

a) Draw a scatter plot of the data set.

Solution:



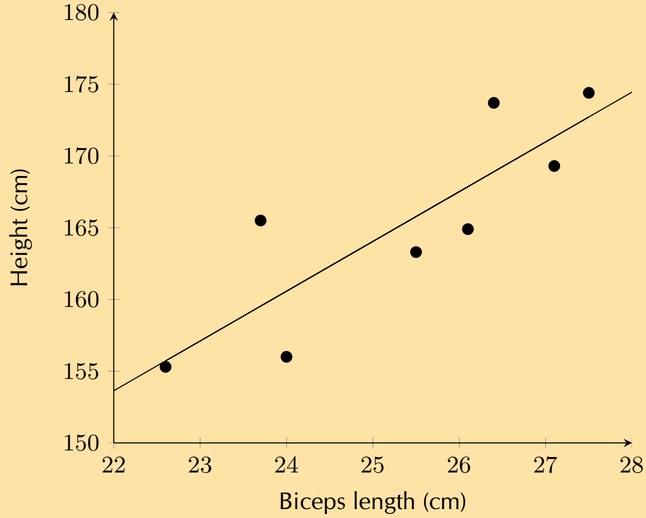
b) Calculate equation of the line of regression.

Solution:

$\hat{y} = 77{,}32 + 3{,}47x$

c) Draw the regression line onto the graph.

Solution:



d) Calculate the correlation coefficient r

Solution:

$r = 0{,}85$

e) What conclusion can you reach, regarding the relationship between the length of the right biceps and height of the students in the data set?

Solution:

The length of the right biceps and the height of the students have a strong, positive linear relationship.

6. A class wrote two tests, and the marks for each were recorded in the table below. Full marks in the first test was 50, and the second test was out of 30.

Learner	**Test 1**	**Test 2**
	(Full marks: 50)	(Full marks: 30)
1	42	25
2	32	19
3	31	20
4	42	26
5	35	23
6	23	14
7	43	24
8	23	12
9	24	14
10	15	10
11	19	11
12	13	10
13	36	22
14	29	17
15	29	17
16	25	16
17	29	18
18	17	
19	30	19
20	28	17

a) Is there a strong correlation between the marks for the first and second test? Show why or why not.

Solution:

Using a calculator, $r = 0{,}98$ which is a very strong, positive, linear correlation between the marks of the first and the second test.

b) One of the learners (in Row 18) did not write the second test. Given her mark for the first test, calculate an expected mark for the second test. Round the mark up to the nearest whole number.

Solution:

Using a calculator, the least squares regression line equation is:

$$\hat{y} = 1{,}08 + 0{,}57x$$

Therefore, the expected mark for the second test for the learner in Row 18 is:

$$y = 1{,}08 + 0{,}57(17) = 10{,}77$$

Therefore the expected mark for the learner in row 18 for the second test is 11 out of 30.

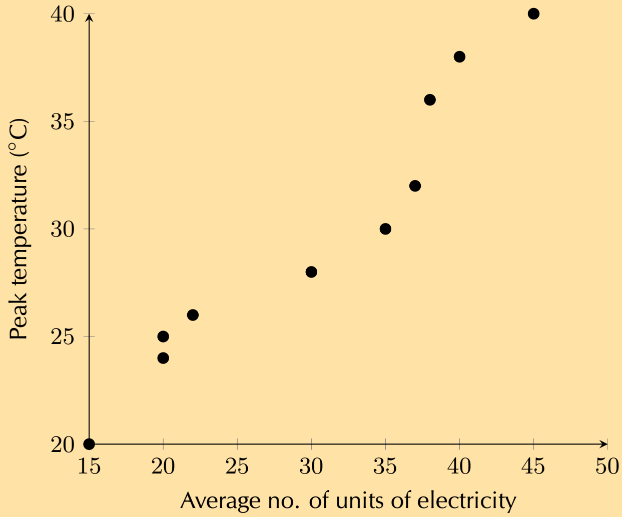
7. Lindiwe works for Eskom, the South African power distributor. She knows that on hot days more electricity than average is used to cool houses. In order to accurately predict how much more electricity needs to be produced, she wants to determine the precise nature of the relationship between temperature and electricity usage.

   The data below shows the peak temperature in degrees Celsius on ten consecutive days during summer and the average number of units of electricity used by a number of households. Examine her data and answer the questions that follow.

Peak temp. (y)	32	40	30	28	25	38	36	20	24	26
Average no. of units (x)	37	45	35	30	20	40	38	15	20	22

a) Draw a scatter plot of the data.

Solution:



b) Using the formulae for a and b, determine the equation of the least squares line.

Solution:

Average no. of units (x)	**Peak temp. (y)**	xy	x^2
37	32	1184	1369
45	40	1800	2025
35	30	1050	1225
30	28	840	900
20	25	500	400
40	38	1520	1600
38	36	1368	1444
15	20	300	225
20	24	480	400
22	26	572	484
$\sum = 302$	$\sum = 299$	$\sum = 9614$	$\sum = 10\,072$

$$b = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n}(x_i)^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$
$$= \frac{10 \times 9614 - 302 \times 299}{10 \times 10\,072 - 302^2} = 0{,}613913409$$

$$a = \bar{y} - b\bar{x} = \frac{299}{10} - 0{,}613913409 \times \frac{302}{10} = 11{,}359815048$$

$$\therefore \hat{y} = 11{,}36 + 0{,}61x$$

c) Determine the value of the correlation coefficient, r, by hand.

Solution:

We have already calculated the value of b by hand in the question above, so we are left to determine σ_x and σ_y.

Average no. of units (x)	**Peak temp. (y)**	$(x-\bar{x})^2$	$(y-\bar{y})^2$
32	37	46,24	4,41
40	45	219,04	102,01
30	35	0,01	23,04
28	30	0,04	3,61
25	20	104,04	24,01
38	40	96,04	65,61
36	38	60,84	37,21
20	15	231,04	98,01
24	20	104,04	34,81
26	22	67,24	15,21
$\sum = 299$	$\sum = 302$	$\sum = 951{,}6$	$\sum = 384{,}9$

$$\sigma_x = \frac{\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}{n} = \frac{\sqrt{951{,}6}}{10} = \pm 3{,}08$$
$$b = 1{,}52$$
$$\sigma_y = \frac{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}}{n} = \frac{\sqrt{384{,}9}}{10} = \pm 1{,}96$$
$$\therefore r = 0{,}61 \times \frac{3{,}08}{1{,}96}$$
$$= 0{,}96$$

d) What can Lindiwe conclude about the relationship between peak temperature and the number of electricity units used?

Solution:

There is a very strong, positive, linear correlation between peak temperature and the average number of electricity units a household uses.

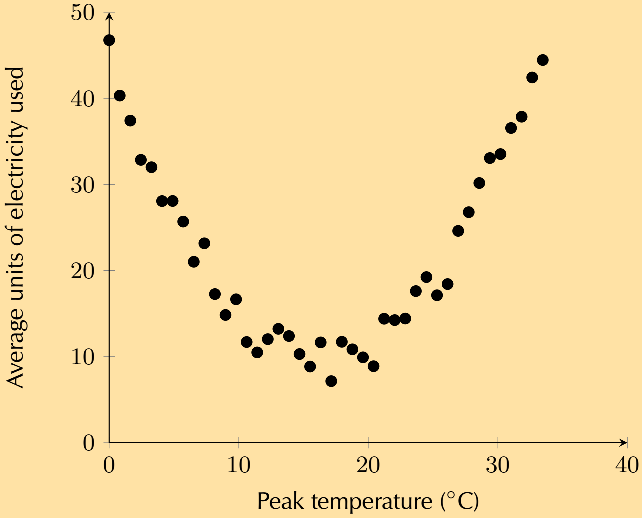
e) Predict the average number of units of electricity used by a household on a day with a peak temperature of $45°$C. Give your answer correct to the nearest unit and identify what this type of prediction is called.

Solution:

$$45 = 11{,}36 + 0{,}61x$$
$$\therefore x = \frac{45 - 11{,}36}{0{,}61}$$
$$= 55{,}15 \approx 55 \text{ units}$$

The value we were asked to predict is outside the range of the available data. This is known as extrapolation.

f) Lindiwe suspected that the relationship between temperature and electricity consumption was not linear for all temperatures. She then decided to collect data for peak temperatures down to $0°$C. Examine the graph of her data below and identify which type of function would best fit the data and describe the nature of the relationship between temperature and electricity for the newly available data.



Solution:

A quadratic function would best fit the data. At about $18°$C average household electricity usage is at its minimum. As the peak temperature gets colder or warmer than this point, electrcity usage increases.

g) Lindiwe is asked by her superiors to determine which day is best to perform maintenance on one of their power plants. She determined that the equation $y = 0{,}13x^2 - 4{,}3x + 45$ best fit her data. Use her equation to estimate the peak temperature and average no. of units used on the day when the least amount of electricity generation is required.

Solution:

This question requires us to find the minimum value of the quadratic equation. There are a number of ways to do this, two are shown below:

The first method is using the formula $x = \frac{-b}{2a}$:

- The first step is to write the equation in the form: $y = ax^2 + bx + c$. Our equation is already in this form, so we can immediately substitute the values into the formula for x.

$$x = \frac{-b}{2a} = \frac{4{,}3}{(2 \times 0{,}13)} = 16{,}54$$

- To find y, we substitute our x-value into the quadratic equation: $0{,}13(16{,}54)^2 - 4{,}3(16{,}54) + 45 = 9{,}44$

Another method is using differentiation:

- The first step is to write the equation in the form: $y = ax^2 + bx + c$. Our equation is already in this form, so we can immediately differentiate the equation.

$$y' = 0{,}13(2)x - 4{,}3 = 0{,}26x + 4{,}3$$

- At the turning point, $y' = 0$, therefore we can now solve for x:

$$0 = 0{,}26x - 4{,}3$$
$$\therefore x = \frac{4{,}3}{0{,}26} = 16{,}54$$

- The x-value can now be substituted into the quadratic equation to find y:

$$y = 0{,}13(16{,}54)^2 - 4{,}3(16{,}54) + 45 = 9{,}44$$

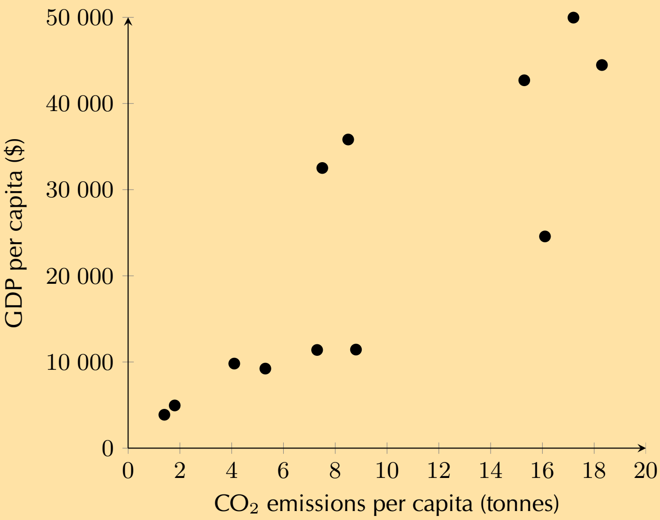
Therefore the peak temperature when electricity demand is at its lowest is $16{,}54°$C and the respective average household electricity usage is $9{,}44$ units.

8. Below is a list of data concerning 12 countries and their respective carbon dioxide (CO_2) emission levels per person per annum (measured in tonnes) and the gross domestic product (GDP is a measure of products produced and services delivered within a country in a year) per person (in US dollars). Data sourced from the World Bank and the US Department of Energy's Carbon Dioxide Information Analysis Center.

	CO_2 emmissions per capita (x)	GDP per capita (y)
South Africa	8,8	11 440
Thailand	4,1	9815
Italy	7,5	32 512
Australia	18,3	44 462
China	5,3	9233
India	1,4	3876
Canada	15,3	42 693
United Kingdom	8,5	35 819
United States	17,2	49 965
Saudi Arabia	16,1	24 571
Iran	7,3	11 395
Indonesia	1,8	4956

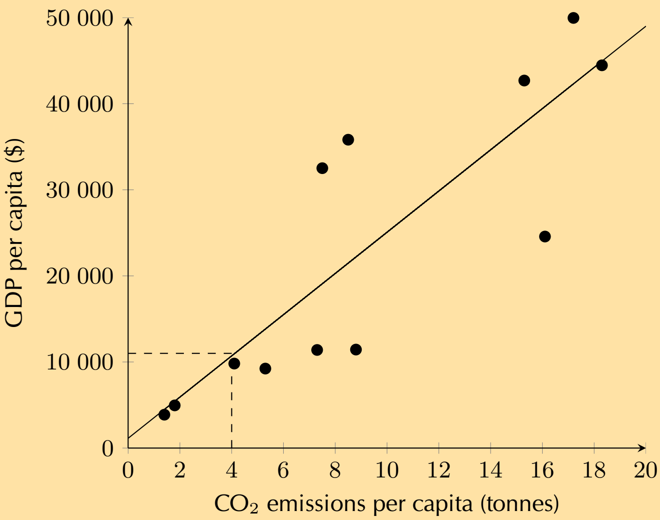
a) Draw a scatter plot of the data set.

Solution:



b) Draw your estimate of the line of best fit on your scatter plot and determine the equation of your line of best fit.

Solution:



The y-intercept is approximately 1000. At $x = 4$, y is approximately 11 000. Therefore, $m = \frac{\Delta y}{\Delta x} = \frac{11000-1000}{4-0} = 2500$

The equation for the line of best fit: $y = 2500x + 1000$

c) Use your calculator to determine the equation for the least squares regression line. Round a and b to two decimal places in your final answer.

Solution:

$a = 1133{,}996106$ and $b = 2393{,}736978$, therefore $\hat{y} = 1134{,}00 + 2393{,}74x$

d) Use your calculator to determine the correlation coefficient, r. Round your answer to two decimal places.

Solution:

$r = 0{,}85$

e) What conclusion can you reach regarding the relationship between CO_2 emissions per annum and GDP per capita for the countries in the data set?

Solution:

There is a strong, positive, linear correlation between CO_2 emissions per annum and GDP per capita for the countries in the data set.

f) Kenya has a GDP per capita of \$ 1712. Use your equation of the least squares regression line to estimate the annual CO_2 emissions of Kenya correct to two decimal places.

Solution:

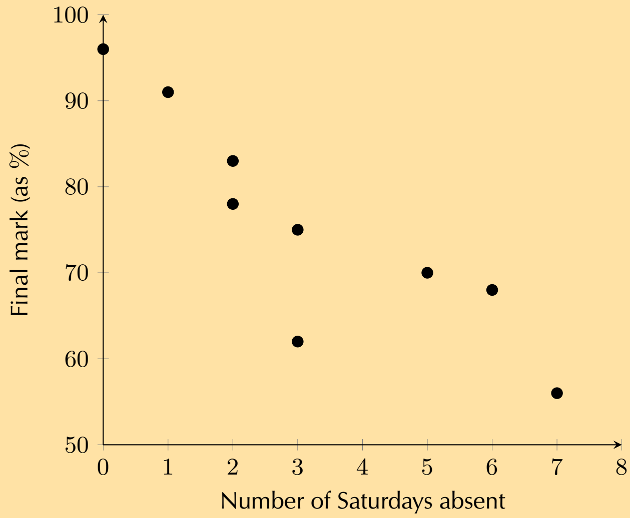
$$1712 = 1134{,}00 + 2393{,}74x$$
$$\therefore x = \frac{1712 - 1134{,}00}{2393{,}74} = 0{,}24 \text{ tonnes}$$

9. A group of students attended a course in Statistics on Saturdays over a period of 10 months. The number of Saturdays on which a student was absent was recorded against the final mark the student obtained. The information is shown in the table below. [Adapted from NSC Paper 3 Feb-March 2012]

Number of Saturdays absent	0	1	2	2	3	3	5	6	7
Final mark (as %)	96	91	78	83	75	62	70	68	56

a) Draw a scatter plot of the data.

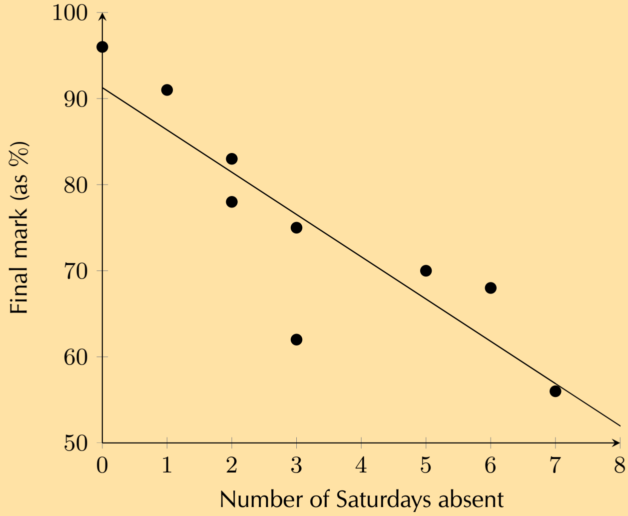
Solution:



b) Determine the equation of the least squares line and draw it on your scatter plot.

Solution:

$$a = 91{,}27$$
$$b = -4{,}91$$
$$\hat{y} = 91{,}27 - 4{,}91x$$



c) Calculate the correlation coefficient.

Solution:

$r = -0{,}87$

d) Comment on the trend of the data.

Solution:

The greater the number of Saturdays absent, the lower the mark.

e) Predict the final mark of a student who was absent for four Saturdays.

Solution:

$$\begin{aligned}\hat{y} &= 91{,}27 - 4{,}91(4)\\ &= 71{,}63\%\\ &\approx 72\%\end{aligned}$$

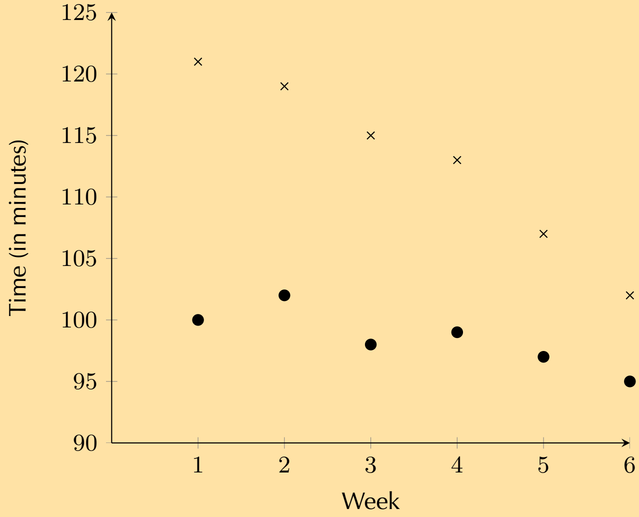
10. Grant and Christie are training for a half-marathon together in 8 weeks time. Christie is much fitter than Grant but she has challenged him to beat her time at the race. Grant has begun a rigid training programme to try and improve his time.

    Time taken to complete a half marathon was recorded each Sunday. The first recorded Sunday is denoted as week 1. The half-marathon takes place on the eighth Sunday, i.e. week 8. Examine the data set in the table below and answer the questions the follow.

Week	1	2	3	4	5	6
Grant's time (HH:MM)	02:01	01:59	01:55	01:53	01:47	01:42
Christie's time (HH:MM)	01:40	01:42	01:38	01:39	01:37	01:35

a) Draw a scatter plot of the data sets. Include Grant and Christie's data on the same set of axes. Use a • to denote Grant's data points and $\times$ to denote Christie's data points. Convert all times to minutes.

Solution:



b) Comment on and compare any trends that you observe in the data.

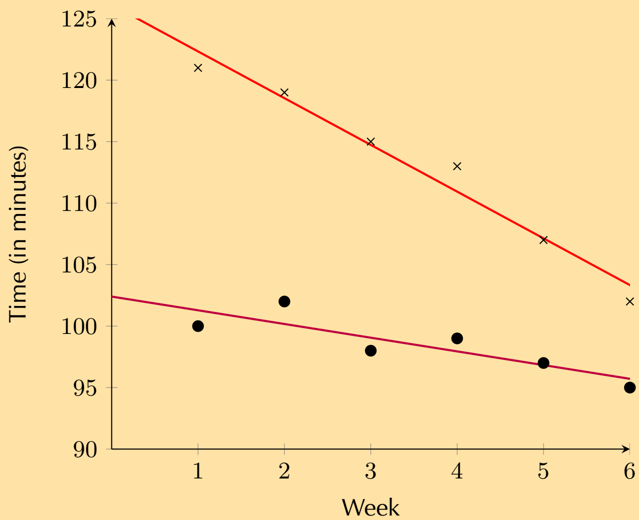
Solution:

Both data sets show negative, linear trends. The trend in Grant's data appears to be more rapidly decreasing than the trend in Christie's data.

c) Determine the equations of the least squares regression lines for Grant's data and Christie's data. Draw these lines on your scatter plot. Use a different colour for each.

Solution:

$$\hat{y}_{\text{Grant}} = 126{,}13 - 3{,}8x$$
$$\hat{y}_{\text{Christie}} = 102{,}4 - 1{,}11x$$



d) Calculate the correlation coefficient and comment on the fit for each data set.

Solution:

Grant: $r = -0{,}98$ (negative, very strong)

Christie: $r = -0{,}86$ (negative, strong)

e) Assuming the observed trends continue, will Grant beat Christie in the race?

Solution:

Grant will beat Christie when $\hat{y}_{\text{Grant}} < \hat{y}_{\text{Christie}}$. To find where the trends intersect, we equate each $\hat{y}$.

$$\begin{aligned}126{,}13 - 3{,}8x &= 102{,}4 - 1{,}11x\\ -3{,}8x + 1{,}11x &= 102{,}4 - 126{,}13\\ -2{,}69x &= -23{,}73\\ x &= 8{,}82\end{aligned}$$

The race takes place in week 8. $8{,}82 > 8$, therefore, Grant will be unable to beat Christie's time when the race takes place.

f) Assuming the observed trends continue, extrapolate the week in which Grant will be able to run a half-marathon in less time than Christie.

Solution:

See answer to e). Grant will be able to beat Christie's time in the ninth week.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29FD 2. 29FF 3. 29FG 4. 29FH 5. 29FJ 6. 29FK
7. 29FM 8. 29FN 9. 29FP 10. 29FQ

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CHAPTER 10

Probability

- This chapter provides good opportunity for experiments and activities in classroom where you can illustrate theoretical probability and number of possible arrangements in practice. Real-life examples have been used extensively in the exercise sections and you may choose to illustrate some of these concepts experimentally in class.
- The terminology and usage of language in this section can be confusing, especially to second-language speakers. Discuss terminology regularly and emphasise the careful reading of questions.
- Union and intersection symbols have been included, but “and” and “or” is the preferred notation in CAPS.
- Make sure to outline the differences between ’and’, ’or’, ’only’ and ’both’. For example, there may be no difference between tea **and** coffee drinkers and tea **or** coffee drinkers in common speech but in probability, the ’and’ and ’or’ have very specific meanings. Tea **and** coffee drinkers refers to the intersection of tea drinkers, i.e. those who drink both beverages, while tea **or** coffee drinkers refers to the union, i.e. those who drink only tea, those who drink only coffee and those who drink both.
- Some learners may find factorial notation challenging. A lengthy problem-set has been provided to try and iron out some common misconceptions, such as $4!3! \neq 12!$ or $\frac{6!}{4!} \neq \frac{3!}{2!}$, but you may have to go over this more slowly with some learners.
- Note that the formula for the arrangement of n different items in r different places i.e. $\frac{n!}{(n-r)!}$ is NOT included in CAPS and learners should therefore be able to solve these problems logically.
- When applying the fundamental counting principle to probability problems, learners may struggle with knowing when to multiply and when to add probabilities. When a number of different outcomes fit a desired result, the probabilities of each outcome are added. When determining the probability of two or more events occurring, their individual probabilities are multiplied.

10.1 Revision

10.2 Identities

Exercise 10 – 1: The product and addition rules

1. Determine whether the following events are dependent or independent and give a reason for your answer:

   a) Joan has a box of yellow, green and orange sweets. She takes out a yellow sweet and eats it. Then, she chooses another sweet and eats it.

      Solution:

      The two events are dependent because there are fewer sweets to choose from when she picks the second time.

   b) Vuzi throws a die twice.

      Solution:

      The two events are independent because the outcome of the first throw has no effect on the outcome of the second throw.

   c) Celia chooses a card at random from a deck of 52 cards. She is unhappy with her choice, so she places the card back in the deck, shuffles it and chooses a second card.

      Solution:

      The two events are independent because the set of cards in the deck is unchanged each time Celia chooses one randomly.

   d) Thandi has a bag of beads. She randomly chooses a yellow bead, looks at it and then puts it back in the bag. Then she randomly chooses another bead and sees that it is red and puts it back in the bag.

Solution:

The two events are independent because there are the same collection of beads each time Thandi chooses one.

e) Mark has a container with calculators. Some of them work and some are broken. He randomly chooses a calculator and sees that it does not work and throws it away. He then chooses another calculator, sees that it works and keeps it.

Solution:

The two events are dependent because Mark has fewer calculators to choose from when he picks again.

2. Given that $P(A) = 0{,}7$; $P(B) = 0{,}4$ and $P(A \text{ and } B) = 0{,}28$,

a) are events A and B mutually exclusive? Give a reason for your answer.

Solution:

For the events to be mutually exclusive $P(A \text{ and } B)$ must be equal to 0. In this case $P(A \text{ and } B) = 0{,}28$, so the events are not mutually exclusive.

b) are the events A and B independent? Give a reason for your answer.

Solution:

For events to be independent: $P(A) \times P(B) = P(A \text{ and } B)$. $P(A) \times P(B) = 0{,}7 \times 0{,}4 = 0{,}28 = P(A \text{ and } B)$. Therefore the events are independent.

3. In the following examples, are A and B dependent or independent?

a) $P(A) = 0{,}2; P(B) = 0{,}7$ and $P(A \text{ and } B) = 0{,}21$

Solution:

$$P(A) \times P(B) = 0{,}2 \times 0{,}7 = 0{,}14 \neq 0{,}21 = P(A \text{ and } B).$$

Therefore the events are dependent.

b) $P(A) = 0{,}2; P(B) = 0{,}7$ and $P(B \text{ and } A) = 0{,}14$.

Solution:

$$P(A) \times P(B) = 0{,}2 \times 0{,}7 = 0{,}14 = P(B \text{ and } A)$$

Therefore the events are independent.

4. $n(A) = 5; n(B) = 4; n(S) = 20$ and $n(A \text{ or } B) = 8$.

a) Are A and B mutually exclusive?

Solution:

$$P(A) = \frac{5}{20}; P(B) = \frac{4}{20}; P(A \text{ or } B) = \frac{8}{20}$$

For A and B to be mutually exclusive: $P(A) + P(B) = P(A \text{ or } B)$.

$$\frac{5}{20} + \frac{4}{20} = \frac{9}{20} \neq \frac{8}{20}$$

The events are therefore not mutually exclusive.

b) Are A and B independent?

Solution:

For A and B to be independent, $P(A) \times P(B) = P(A \text{ and } B)$

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ \therefore P(A \text{ and } B) &= P(A) + P(B) - P(A \text{ or } B) \\ &= \frac{5}{20} + \frac{4}{20} - \frac{8}{20} \\ &= \frac{1}{20} \\ P(A) \times P(B) &= \frac{1}{4} \times \frac{1}{5} \\ &= \frac{1}{20} = P(A \text{ and } B) \end{aligned}$$

Therefore the events are independent.

5. Simon rolls a die twice. What is the probability of getting:

a) two threes.

Solution:

$$P(\text{two threes}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

b) a prime number then an even number.

Solution:

There are 3 possible prime numbers on a die, namely, 2, 3, and 5, and there are 3 possible even numbers, namely, 2, 4, and 6.

$$\begin{aligned} P(\text{prime number then even number}) &= P(\text{prime number}) \times P(\text{even number}) \\ &= \frac{3}{6} \times \frac{3}{6} = \frac{9}{36} = \frac{1}{4} \end{aligned}$$

c) no threes.

Solution:

If no threes are rolled then, for each of the events, 5 possibilities remain.

$$P(\text{no threes}) = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$$

d) only one three.

Solution:

In the sample space there are 36 possible outcomes. There are two ways of getting only one three: either getting a 3 on the first throw and a number other than three on the second throw or a 3 on the second throw and a number other than 3 on the first throw. The outcomes containing only one 3 are: $(3;1)$; $(3;2)$; $(3;4)$; $(3;5)$; $(3;6)$; $(1;3)$; $(2;3)$; $(4;3)$; $(5;3)$; $(6;3)$.

$$P(\text{only one } 3) = \frac{10}{36} = \frac{5}{18}$$

e) at least one three.

Solution:

In the sample space there are 36 possible outcomes. The outcomes containing at least one 3 are: $(3;1)$; $(3;2)$; $(3;3)$; $(3;4)$; $(3;5)$; $(3;6)$; $(1;3)$; $(2;3)$; $(4;3)$; $(5;3)$; $(6;3)$.

$$P(\text{at least one } 3) = \frac{11}{36}$$

6. The Mandalay Secondary soccer team has to win both of their next two matches in order to qualify for the finals. The probability that Mandalay Secondary will win their first soccer match against Ihlumelo High is $\frac{2}{5}$ and the probability of winning their second soccer match against Masiphumelele Secondary is $\frac{3}{7}$. Assume each match is an independent event.

a) What is the probability they will progress to the finals?

Solution:

$$\begin{aligned} P(\text{win and win}) &= \frac{2}{5} \times \frac{3}{7} \\ &= \frac{6}{35} \end{aligned}$$

b) What is the probability they will not win either match?

Solution:

To calculate the probability of not winning a match use:

$$P(\text{not win}) = 1 - P(\text{win})$$

$$\text{Therefore } P(\text{not win and not win}) = \frac{3}{5} \times \frac{4}{7} = \frac{12}{35}$$

This solution makes use of the complementary rule which students should be familiar with. We will revise the rule in more detail later.

c) What is the probability they will win only one of their matches?

Solution:

There are two possible outcomes: win-not win or not win-win. Let win $= W$.

$$\begin{aligned} P((\text{W;not W}) \text{ or } (\text{not W;W})) &= P(\text{W;not W}) + P(\text{not W;W}) \\ &= P(\text{W}) \times P(\text{not W}) + P(\text{not W}) \times P(\text{W}) \\ &= \frac{2}{5} \times \frac{4}{7} + \frac{3}{5} \times \frac{3}{7} \\ &= \frac{8}{35} + \frac{9}{35} = \frac{17}{35} \end{aligned}$$

d) You were asked to assume that the matches are independent events but this is unlikely in reality. What are some factors you think may result in the outcome of the matches being dependent?

Solution:

This is an open-ended question designed to get learners to think critically about the dependent or independent nature of real life events. Example answers could include injuries to or suspensions of players during the first match, team morale if they win or lose the first match, etc.

7. A pencil bag contains 2 red pens and 4 green pens. A pen is drawn from the bag and then replaced before a second pen is drawn. Calculate:

a) The probability of drawing a red pen first if a green pen is drawn second.

Solution:

The events are independent so:

$$P(\text{red pen first}) = \frac{2}{6} = \frac{1}{3}$$

b) The probability of drawing a green pen second if the first pen drawn was red.

Solution:

The events are independent so:

$$P(\text{green pen second}) = \frac{4}{6} = \frac{2}{3}$$

c) The probability of drawing a red pen first and a green pen second.

Solution:

$$\begin{aligned} P(\text{first pen red and second pen green}) &= \frac{1}{3} \times \frac{2}{3} \\ &= \frac{2}{9} \end{aligned}$$

8. A lunch box contains 4 sandwiches and 2 apples. Vuyele chooses a food item randomly and eats it. He then chooses another food item randomly and eats that. Determine the following:

a) The probability that the first item is a sandwich.

Solution:

$$P(\text{sandwich first}) = \frac{4}{6} = \frac{2}{3}$$

b) The probability that the first item is a sandwich and the second item is an apple.

Solution:
First item sandwich and second item apple (SA):

$$\frac{4}{6} \times \frac{2}{5} = \frac{8}{30} = \frac{4}{15}$$

c) The probability that the second item is an apple.

Solution:
There are two possible outcomes of getting an apple second:

- first item sandwich and second item apple (SA):

$$= \frac{4}{15} \text{ (from b)}$$

- first item apple and second item apple (AA):

$$\frac{2}{6} \times \frac{1}{5} = \frac{2}{30} = \frac{1}{15}$$

$$\begin{aligned} P(\text{apple second}) &= P(SA) + P(AA) = \frac{4}{15} + \frac{1}{15} \\ &= \frac{1}{3} \end{aligned}$$

d) Are the events in a) and c) dependent? Confirm your answer with a calculation.

Solution:

$$P(\text{sandwich first}) \times P(\text{apple second}) = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9} \neq \frac{4}{15} = P(\text{SA})$$

Therefore the events are dependent.

9. Given that $P(A) = 0{,}5$; $P(B) = 0{,}4$ and $P(A \text{ or } B) = 0{,}7$, determine by calculation whether events A and B are:

a) mutually exclusive

Solution:

$$P(A) + P(B) = 0{,}5 + 0{,}4 = 0{,}9 \neq 0{,}7 = P(A \text{ or } B)$$

Therefore, A and B are not mutually exclusive.

b) independent

Solution:

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ 0{,}7 &= 0{,}5 + 0{,}4 - P(A \text{ and } B) \\ P(A \text{ and } B) &= 0{,}5 + 0{,}4 - 0{,}7 = 0{,}2 \\ P(A) \times P(B) &= 0{,}5 \times 0{,}4 = 0{,}2 = P(A \text{ and } B) \end{aligned}$$

Therefore A and B are independent.

10. A and B are two events in a sample space where $P(A) = 0{,}3$; $P(A \text{ or } B) = 0{,}8$ and $P(B) = k$. Determine the value of k if:

a) A and B are mutually exclusive

Solution:
For A and B to be mutually exclusive: $P(A) + P(B) = P(A \text{ or } B)$

$$\begin{aligned} 0{,}3 + k &= 0{,}8 \\ \therefore k &= 0{,}5 \end{aligned}$$

b) A and B are independent

Solution:
For A and B to be independent:

$$P(A \text{ and } B) = P(A) \times P(B)$$
$$\text{Therefore } P(A \text{ and } B) = 0{,}3k$$

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ 0{,}8 &= 0{,}3 + k - 0{,}3k \\ 0{,}8 &= 0{,}3 + 0{,}7k \\ \therefore 0{,}7k &= 0{,}5 \\ \therefore k &= \frac{5}{7} \end{aligned}$$

11. A and B are two events in sample space S where $n(S) = 36$; $n(A) = 9$; $n(B) = 4$ and $n(\text{not } (A \text{ or } B)) = 24$. Determine:

    a) $P(A \text{ or } B)$
    Solution:

$$\begin{aligned} P(A \text{ or } B) &= 1 - P(\text{not } (A \text{ or } B)) \\ &= 1 - \frac{24}{36} = \frac{1}{3} \end{aligned}$$

    b) $P(A \text{ and } B)$
    Solution:

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ \frac{1}{3} &= \frac{9}{36} + \frac{4}{36} - P(A \text{ and } B) \\ \therefore P(A \text{ and } B) &= \frac{9}{36} + \frac{4}{36} - \frac{1}{3} \\ &= \frac{1}{36} \end{aligned}$$

    c) whether events A and B independent. Justify your answer with a calculation.
    Solution:
    For independent events $P(A) \times P(B) = P(A \text{ and } B)$

$$P(A) \times P(B) = \frac{1}{4} \times \frac{1}{9} = \frac{1}{36} = P(A \text{ and } B)$$

    Therefore A and B are independent.

12. The probability that a Mathematics teacher is absent from school on a certain day is $0{,}2$. The probability that the Science teacher will be absent that same day is $0{,}3$.

    a) Do you think these two events are independent? Give a reason for your answer.
    Solution:
    Learner dependent. For example: No, there could a bug or illness spreading through the school, therefore the absence of both teachers may be dependent.

    b) Assuming the events are independent, what is the probability that the Mathematics teacher or the Science teacher is absent?
    Solution:
    Let the probability that the Mathematics teacher is absent $= P(M)$ and the probability that the Science teacher is absent $= P(S)$.

$$P(M \text{ or } S) = P(M) + P(S) - P(M \text{ and } S)$$

Assuming the events are independent:

$$\begin{aligned} P(M \text{ and } S) &= 0{,}2 \times 0{,}3 = 0{,}06 \\ \text{Therefore } P(M \text{ or } S) &= 0{,}2 + 0{,}3 - 0{,}06 \\ &= 0{,}44 \end{aligned}$$

c) What is the probability that neither the Mathematics teacher nor the Science teacher is absent?

Solution:

$$P(\text{not } (M \text{ or } S)) = 1 - 0{,}44 = 0{,}56$$

13. Langa Cricket Club plays two cricket matches against different clubs. The probability of winning the first match is $\frac{3}{5}$ and the probability of winning the second match is $\frac{4}{9}$. Assuming the results of the matches are independent, calculate the probability that Langa Cricket Club will:

a) win both matches.

Solution:

Let $P(M)$ = the probability of winning the first match and $P(N)$ = the probability of winning the second match.

$$\begin{aligned} P(M \text{ and } N) &= \frac{3}{5} \times \frac{4}{9} \\ &= \frac{12}{45} \\ &= \frac{4}{15} \end{aligned}$$

b) not win the first match.

Solution:

$$\begin{aligned} P(\text{not } M) &= 1 - \frac{3}{5} \\ &= \frac{2}{5} \end{aligned}$$

c) win one or both of the two matches.

Solution:

$$\begin{aligned} P(M \text{ or } N) &= P(M) + P(N) - P(M \text{ and } N) \\ &= \frac{3}{5} + \frac{4}{9} - \frac{4}{15} \\ &= \frac{7}{9} \end{aligned}$$

d) win neither match.

Solution:

$$\begin{aligned} P(\text{not } M \text{ and not } N) &= P(\text{not } M) \times P(\text{not } N) \\ &= \left(1 - \frac{3}{5}\right) \times \left(1 - \frac{4}{9}\right) \\ &= \frac{2}{5} \times \frac{5}{9} \\ &= \frac{2}{9} \end{aligned}$$

e) not win the first match and win the second match.

Solution:

$$\begin{aligned} P(\text{not } M \text{ and } N) &= P(\text{not } M) \times P(N) \\ &= \frac{2}{5} \times \frac{4}{9} \\ &= \frac{8}{45} \end{aligned}$$

14. Two teams are working on the final problem at a Mathematics Olympiad. They have 10 minutes remaining to finish the problem. The probability that team A will finish the problem in time is 40% and the probability that team B will finish the problem in time is 25%. Calculate the probability that both teams will finish before they run out of time.

Solution:

Let the probability that team A will finish $= P(A)$ and the probability team B will finish $= P(B)$. The teams are working separately, therefore the two events are independent.

$$\begin{aligned} P(A \text{ and } B) &= P(A) \times P(B) \\ &= 0.4 \times 0.25 \\ &= 0.1 \text{ or } 10\% \end{aligned}$$

15. Thabo and Julia were arguing about whether people prefer tea or coffee. Thabo suggested that they do a survey to settle the dispute. In total, they surveyed 24 people and found that 8 of them preferred to drink coffee and 12 of them preferred to drink tea. The number of people who drink tea, coffee or both is 16. Determine:

a) the probability that a person drinks tea, coffee or both.

Solution:

Let $n(C)$ be the number of people who drink coffee and $n(T)$ be the number of people who drink tea.

$$\begin{aligned} P(C \text{ or } T) &= \frac{n(C \text{ or } T)}{n(S)} \\ &= \frac{16}{24} \\ &= \frac{2}{3} \end{aligned}$$

b) the probability that a person drinks neither tea nor coffee.

Solution:

$$\begin{aligned} P(\text{not } (C \text{ or } T)) &= 1 - P(C \text{ or } T) \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3} \end{aligned}$$

c) the probability that a person drinks coffee and tea.

Solution:

$$\begin{aligned} P(C \text{ and } T) &= P(C) + P(T) - P(C \text{ or } T) \\ &= \frac{n(C)}{n(S)} + \frac{n(T)}{n(S)} - \frac{n(C \text{ or } T)}{n(S)} \\ &= \frac{8}{24} + \frac{12}{24} - \frac{16}{24} \\ &= \frac{1}{6} \end{aligned}$$

d) the probability that a person does not drink coffee.

Solution:

$$\begin{aligned} P(\text{not } C) &= 1 - P(C) \\ &= 1 - \frac{8}{24} \\ &= \frac{2}{3} \end{aligned}$$

e) whether the event that a person drinks coffee and the event that a person drinks tea are independent.

Solution:

$$\begin{aligned} P(C) \times P(T) &= \frac{1}{3} \times \frac{1}{2} \\ &= \frac{1}{6} = P(C \text{ and } T) \end{aligned}$$

Therefore the events are independent.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29FV 2. 29FW 3. 29FX 4. 29FY 5. 29FZ 6. 29G2
7. 29G3 8. 29G4 9. 29G5 10. 29G6 11. 29G7 12. 29G8
13. 29G9 14. 29GB 15. 29GC

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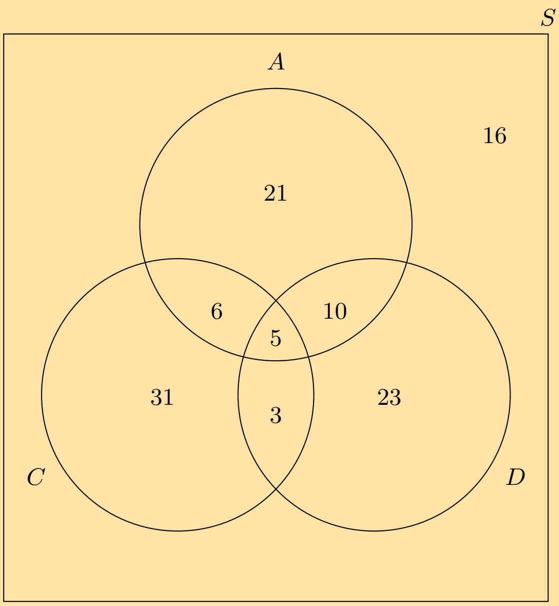
10.3 Tools and Techniques

Exercise 10 – 2: Venn and tree diagrams

1. A survey was done on a group of learners to determine which type of TV shows they enjoy: action, comedy or drama. Let A = action, C = comedy and D = drama. The results of the survey are shown in the Venn diagram below.



Study the Venn diagram and determine the following:

a) the total number of learners surveyed

Solution:

115

b) the number of learners who do not enjoy any of the mentioned types of TV shows

Solution:

16

c) $P(\text{not } A)$

Solution:

$\frac{73}{115}$

d) $P(A \text{ or } D)$

Solution:

$\frac{68}{115}$

e) $P(A \text{ and } C \text{ and } D)$

Solution:

$\frac{5}{115} = \frac{1}{23}$

f) $P(\text{not } (A \text{ and } D))$

Solution:

$\frac{100}{115} = \frac{20}{23}$

g) $P(A \text{ or not } C)$

Solution:

$\frac{81}{115}$

h) $P(\text{not } (A \text{ or } C))$

Solution:

$\frac{39}{115}$

i) the probability of a learner enjoying at least two types of TV shows

Solution:

$\frac{24}{115}$

j) Describe, in words, the meaning of each of the questions c) to h) in the context of this problem.

Solution:

$P(\text{not } A)$: the probability that learners do not enjoy action TV shows

$P(A \text{ or } D)$: the probability that learners enjoy action or drama TV shows

$P(A \text{ and } D \text{ and } C)$: the probability that learners enjoy action, drama and comedy TV shows

$P(\text{not } (A \text{ and } D))$: the probability that learners do not enjoy action and drama TV shows

$P(A \text{ or not } C)$: the probability that learners enjoy action TV shows or do not enjoy comedy TV shows

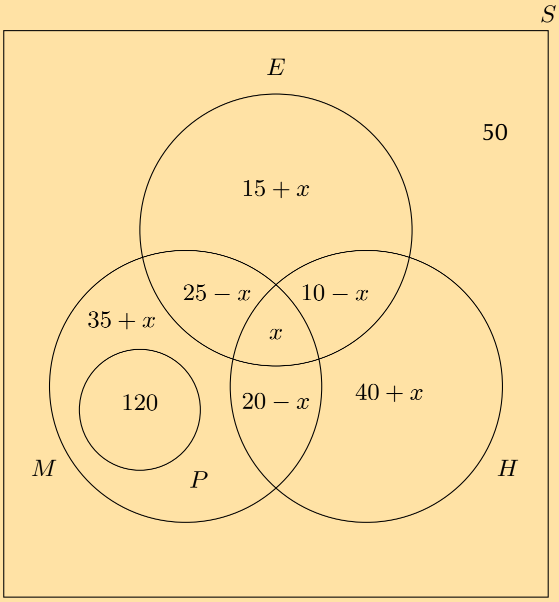
$P(\text{not } (A\text{or } C))$: the probability that learners do not enjoy action or comedy TV shows

2. At Thandokulu Secondary School, there are are 320 learners in Grade 12, 270 of whom take one or more of Mathematics, History and Economics. The subject choice is such that everybody who takes Physical Sciences must also take Mathematics and nobody who takes Physical Sciences can take History or Economics. The following is known about the number of learners who take these subjects:

- 70 take History
- 50 take Economics
- 120 take Physical Sciences
- 200 take Mathematics
- 20 take Mathematics and History
- 10 take History and Economics
- 25 take Mathematics and Economics
- x learners take Mathematics and History and Economics

a) Represent the information above in a Venn diagram. Let Mathematics be M, History be H, Physical Sciences be P and Economics be E.

Solution:



b) Determine the number of learners, x, who take Mathematics, History and Economics.

Solution:

$$120 + (35 + x) + (25 - x) + (20 - x) + x + (40 + x) + (10 - x) + (15 + x) = 270$$
$$265 + x = 270$$
$$x = 5$$

Therefore 5 learners take Mathematics, History and Economics.

c) Determine $P(\text{not } (M \text{ or } H \text{ or } E))$ and state in words what your answer means.

Solution:

$$P(\text{not } (M \text{ or } H \text{ or } E)) = \frac{50}{320} = \frac{5}{32}$$

This is the probability that a learner does not take Mathematics, History or Economics.

d) Determine the probability that a learner takes at least two of these subjects.

Solution:

This question requires us to find the sum of the probabilities of all the learners who take at least two subjects. This includes the intersection of each of the subjects.

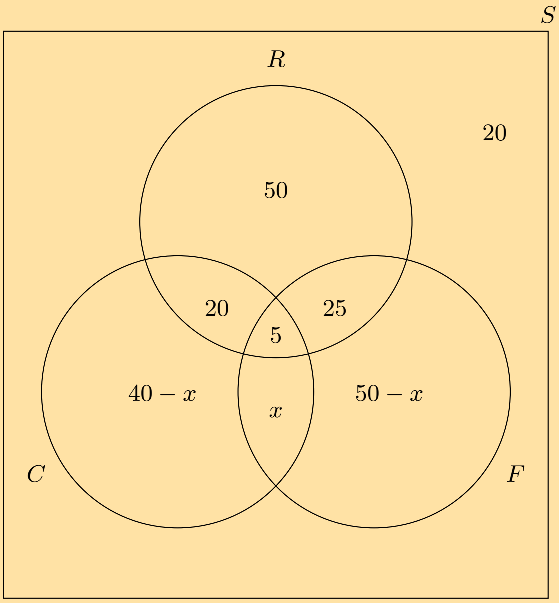
$$P(\text{at least two subjects}) = \frac{120 + 20 + 5 + 5 + 15}{320}$$
$$= \frac{33}{64}$$

3. A group of 200 people were asked about the kind of sports they watch on television. The information collected is given below:

- 180 watch rugby, cricket or soccer
- 5 watch rugby, cricket and soccer
- 25 watch rugby and cricket
- 30 watch rugby and soccer
- 100 watch rugby
- 65 watch cricket
- 80 watch soccer
- x watch cricket and soccer but not rugby

a) Represent all the above information in a Venn diagram. Let rugby watchers $= R$, cricket watchers $= C$ and soccer watchers $= F$.

Solution:



b) Find the value of x.

Solution:

$$\begin{aligned} 50 + 25 + 5 + 20 + (40 - x) + (50 - x) + x &= 180 \\ 190 - x &= 180 \\ \text{Therefore } x &= 10 \end{aligned}$$

c) Determine $P(\text{not } (R \text{ or } F \text{ or } C))$

Solution:

$$P(\text{not } (R \text{ or } F \text{ or } C)) = \frac{20}{200} = \frac{1}{10}$$

d) Determine $P(R \text{ or } F \text{ or not } C)$

Solution:

$$P(R \text{ or } F \text{ or not } C) = \frac{170}{200} = \frac{17}{20}$$

e) Are watching cricket and watching rugby independent events? Confirm your answer using a calculation.

Solution:

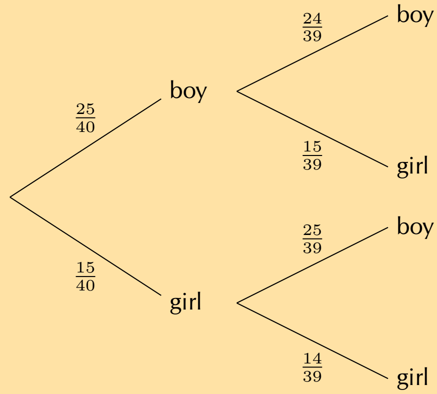
$$\begin{aligned} P(R) &= \frac{100}{200} = \frac{1}{2} \\ P(C) &= \frac{65}{200} = \frac{13}{40} \\ P(R \text{ and } C) &= \frac{25}{200} = \frac{1}{8} \\ P(R) \times P(C) &= \frac{1}{2} \times \frac{13}{40} = \frac{13}{80} \neq \frac{1}{8} = P(R \text{ and } C) \end{aligned}$$

Therefore watching rugby and watching cricket are dependent events.

4. There are 25 boys and 15 girls in the English class. Each lesson, two learners are randomly chosen to do an oral.

   a) Represent the composition of the English class in a tree diagram. Include all possible outcomes and probabilities.

Solution:



b) Calculate the probability that a boy and a girl are chosen to do an oral in any particular lesson.

Solution:

$$\begin{aligned}\left(\frac{25}{40}\times\frac{15}{39}\right)+\left(\frac{15}{40}\times\frac{25}{39}\right) &= \frac{25}{104}+\frac{25}{104}\\ &= \frac{25}{52}\end{aligned}$$

c) Calculate the probability that at least one of the learners chosen to do an oral in any particular lesson is male.

Solution:

Note: This question can be answered by subtracting the outcome not containing a boy (girl; girl) from 1 (shown below) or by adding the three outcomes which include a boy. Either method is correct.

$$\begin{aligned}1-\left(\frac{15}{40}\times\frac{14}{39}\right) &= 1-\frac{7}{52}\\ &= \frac{45}{52}\end{aligned}$$

d) Are the events picking a boy first and picking a girl second independent or dependent? Justify your answer with a calculation.

Solution:

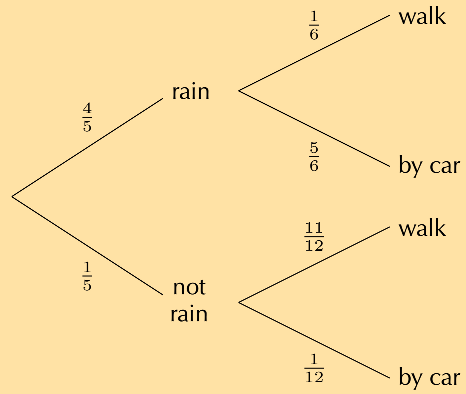
$$\begin{aligned}P(\text{boy first}) &= \frac{25}{40}=\frac{5}{8}\\ P(\text{girl second}) &= \left(\frac{15}{40}\times\frac{14}{39}\right)+\left(\frac{25}{40}\times\frac{15}{39}\right)\\ &= \frac{7}{52}+\frac{25}{104}\\ &= \frac{3}{8}\\ P(\text{boy first})\times P(\text{girl second}) &= \frac{5}{8}\times\frac{3}{8}=\frac{15}{64}\\ P(\text{boy first and girl second}) &= \frac{25}{40}\times\frac{15}{39}\\ &= \frac{25}{104}\neq\frac{15}{64}=P(\text{boy first})\times P(\text{girl second})\end{aligned}$$

Therefore picking a boy first and picking a girl second are dependent events.

5. During July in Cape Town, the probability that it will rain on a randomly chosen day is $\frac{4}{5}$. Gladys either walks to school or gets a ride with her parents in their car. If it rains, the probability that Gladys' parents will take her to school by car is $\frac{5}{6}$. If it does not rain, the probability that Gladys' parents will take her to school by car is $\frac{1}{12}$.

a) Represent the above information in a tree diagram. On your diagram show all the possible outcomes and respective probabilities.

Solution:



b) What is the probability that it is a rainy day and Gladys walks to school?

Solution:

$$\begin{aligned} P(\text{rain and walk}) &= \frac{4}{5} \times \frac{1}{6} \\ &= \frac{2}{15} \end{aligned}$$

c) What is the probability that Gladys' parents take her to school by car?

Solution:

$$\begin{aligned} P(\text{by car}) &= P(\text{rain and car}) + P(\text{no rain and car}) \\ &= \left(\frac{4}{5} \times \frac{5}{6}\right) + \left(\frac{1}{5} \times \frac{1}{12}\right) \\ &= \frac{2}{3} + \frac{1}{60} \\ &= \frac{41}{60} \end{aligned}$$

6. There are two types of property burglaries: burglary of private residences and burglary of business premises. In Metropolis, burglary of a private residence is four times as likely as that of a business premises. The following statistics for each type of burglary were obtained from the Metropolis Police Department:

Burglary of private residences

Following a burglary:

- 25% of criminals are arrested within 48 hours.
- 15% of criminals are arrested after 48 hours.
- 60% of criminals are never arrested for that particular burglary.

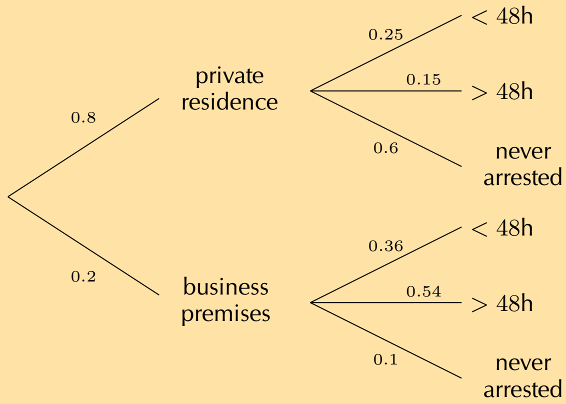
Burglary of business premises

Following a burglary:

- 36% of criminals are arrested within 48 hours.
- 54% of criminals are arrested after 48 hours.
- 10% of criminals are never arrested for that particular burglary.

a) Represent the information above in a tree diagram, showing all outcomes and respective probabilities.

Solution:



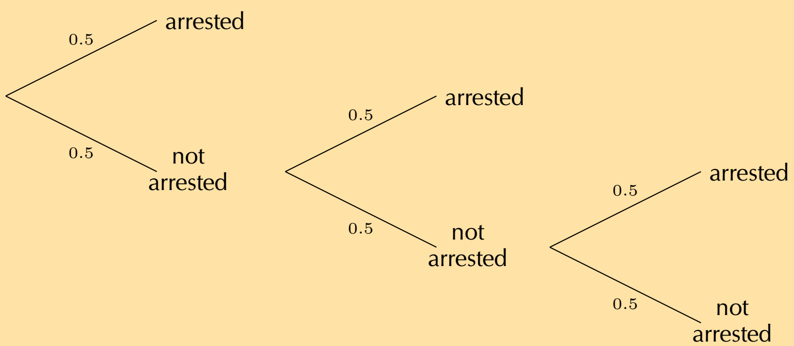
b) Calculate the probability that a private home is burgled and nobody is arrested.
Solution:

$$\begin{aligned} P(\text{private home and never arrested}) &= 0{,}8 \times 0{,}6 \\ &= 0{,}48 \end{aligned}$$

c) Calculate the probability that burglars of private homes and business premises are arrested.
Solution:

$$\begin{aligned} P(\text{arrested}) &= (0{,}8 \times 0{,}25) + (0{,}8 \times 0{,}15) + (0{,}2 \times 0{,}36) + (0{,}2 \times 0{,}54) \\ &= 0{,}5 \end{aligned}$$

d) Use your answer in the previous question to construct a tree diagram to calculate the probability that a burglar is arrested after at most three burglaries.
Solution:



$$\begin{aligned} P(\text{arrested after 3 burglaries}) &= 0{,}5 + (0{,}5 \times 0{,}5) + (0{,}5 \times 0{,}5 \times 0{,}5) \\ &= 0{,}875 \end{aligned}$$

This answer could also be reached by subtracting the probability of not being arrested after three burglaries from 1:

$$1 - (0{,}5 \times 0{,}5 \times 0{,}5) = 1 - 0{,}5^3 = 1 - 0{,}125 = 0{,}875$$

We will use this principle to answer the next question.

e) Determine after how many burglaries a burglar has at least a
   i. 90% chance of being arrested.
   ii. 99% chance of being arrested.

Solution:
Let the number of burglaries $= n$

i.

$$\begin{aligned} 0{,}90 &= 1 - P(\text{not arrested})^n \\ &= 1 - 0{,}5^n \\ \text{Therefore } 0{,}1 &= 0{,}5^n \\ \text{Therefore } n &= \log_{0{,}5} 0{,}1 \\ &= 3{,}32 \end{aligned}$$

After 4 burglaries, there will be at least a 90% chance of being arrested.

ii.

$$\begin{aligned} 0{,}99 &= 1 - P(\text{not arrested})^n \\ &= 1 - 0{,}5^n \\ \text{Therefore } 0{,}01 &= 0{,}5^n \\ \text{Therefore } n &= \log_{0{,}5} 0{,}01 \\ &= 6{,}64 \end{aligned}$$

After 7 burglaries, there will be at least a 99% chance of being arrested.

Check answers online with the exercise code below or click on 'show me the answer'.
1. 29GF 2. 29GG 3. 29GH 4. 29GJ 5. 29GK 6. 29GM

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Exercise 10 – 3: Contingency tables

1. A number of drivers were asked about the number of motor vehicle accidents they were involved in over the last 10 years. Part of the data collected is shown in the table below.

	≤ 2 accidents	**> 2 accidents**	**Total**
Female	210	90	
Male			
Total	350	150	500

a) What are the variables investigated here and what is the purpose of the research?

Solution:

The variables are gender and number of accidents over a period of 10 years. The purpose of the research is to determine if gender is related to the number of accidents a driver is involved in.

b) Complete the table.

Solution:

	≤ 2 accidents	**> 2 accidents**	**Total**
Female	210	90	300
Male	140	60	200
Total	350	150	500

c) Determine whether gender and number of accidents are independent suing a calculation.

Solution:

$$P(\text{female}) = \frac{300}{500} = 0{,}6$$
$$P(\text{male}) = \frac{200}{500} = 0{,}4$$
$$P(\leq 2 \text{ accidents}) = \frac{350}{500} = 0{,}7$$
$$P(> 2 \text{ accidents}) = \frac{150}{500} = 0{,}3$$
$$P(\text{female and } \leq 2 \text{ accidents}) = \frac{210}{500} = 0{,}42$$
$$P(\text{female and } > 2 \text{ accidents}) = \frac{90}{500} = 0{,}18$$
$$P(\text{male and } \leq 2 \text{ accidents}) = \frac{140}{500} = 0{,}28$$
$$P(\text{male and } > 2 \text{ accidents}) = \frac{60}{500} = 0{,}12$$
$$P(\text{female}) \times P(\leq 2 \text{ accidents}) = 0{,}42 = P(\text{female and } \leq 2 \text{ accidents})$$
$$P(\text{female}) \times P(> 2 \text{ accidents}) = 0{,}18 = P(\text{female and } > 2 \text{ accidents})$$
$$P(\text{male}) \times P(\leq 2 \text{ accidents}) = 0{,}28 = P(\text{male and } \leq 2 \text{ accidents})$$
$$P(\text{male}) \times P(> 2 \text{ accidents}) = 0{,}12 = P(\text{male and } > 2 \text{ accidents})$$

It can be seen that in all cases $P(A) \times P(B) = P(A \text{ and } B)$, therefore number of motor vehicle accidents is independent of the gender of the driver.

2. Researchers conducted a study to test how effective a certain inoculation is at preventing malaria. Part of their data is shown below:

	Malaria	**No malaria**	**Total**
Male	a	b	216
Female	c	d	648
Total	108	756	864

a) Calculate the probability that a randomly selected study participant will be female.

Solution:

$P(\text{female}) = \frac{648}{864} = \frac{3}{4}$

b) Calculate the probability that a randomly selected study participant will have malaria.

Solution:

$P(\text{malaria}) = \frac{108}{864} = \frac{1}{8}$

c) If being female and having malaria are independent events, calculate the value c.

Solution:

$$P(\text{female and malaria}) = \frac{3}{4} \times \frac{1}{8} = \frac{3}{32}$$
$$\therefore c = \frac{3}{32} \times 864 = 81$$

d) Using the value of c, fill in the missing values on the table.

Solution:

	Malaria	**No malaria**	**Total**
Male	27	189	216
Female	81	567	648
Total	108	756	864

3. The reaction time of 400 drivers during an emergency stop was tested. Within the study cohort (the group of people being studied), the probability that a driver chosen at random was 40 years old or younger is 0,3 and the probability of a reaction time less than 1,5 seconds is 0,7.

a) Calculate the number of drivers who are 40 years old or younger.

Solution:

$n(\text{forty years and younger}) = 0{,}3 \times 400 = 120$

b) Calculate the number of drivers who have a reaction time of less than $1{,}5$ seconds.

Solution:

$n(\text{reaction time } < 1{,}5 \text{ s}) = 0{,}7 \times 400 = 280$

c) If age and reaction time are independent events, calculate the number of drivers 40 years old and younger with a reaction time of less than $1{,}5$ seconds.

Solution:

$$P(\text{40 or younger and reaction time } < 1{,}5 \text{ secs}) = 0{,}3 \times 0{,}7 = 0{,}21$$
$$\therefore n(\text{40 or younger and reaction time } < 1{,}5 \text{ secs}) = 0{,}21 \times 400 = 84$$

d) Complete the table below.

	Reaction time $< 1{,}5$ s	**Reaction time $> 1{,}5$ s**	**Total**
≤ 40 years			
> 40 years			
Total			400

Solution:

	Reaction time $< 1{,}5$ s	**Reaction time $> 1{,}5$ s**	**Total**
≤ 40 years	84	36	120
> 40 years	196	84	280
Total	280	120	400

4. A new treatment for influenza (the flu) was tested on a number of patients to determine if it was better than a placebo (a pill with no therapeutic value). The table below shows the results three days after treatment:

	Flu	**No flu**	**Total**
Placebo	228	60	
Treatment			
Total	240	312	

a) Complete the table.

Solution:

	Flu	**No flu**	**Total**
Placebo	228	60	288
Treatment	12	252	264
Total	240	312	552

b) Calculate the probability of a patient receiving the treatment.

Solution:

$$\begin{aligned} P(\text{treatment}) &= \frac{n(\text{treatment})}{n(\text{total patients})} \\ &= \frac{312}{552} = \frac{13}{23} \end{aligned}$$

c) Calculate the probability of a patient having no flu after three days.

Solution:

$$\begin{aligned} P(\text{no flu}) &= \frac{n(\text{no flu})}{n(\text{total patients})} \\ &= \frac{264}{552} = \frac{11}{23} \end{aligned}$$

d) Calculate the probability of a patient receiving the treatment and having no flu after three days.

Solution:

$$\begin{aligned} P(\text{no flu and treatment}) &= \frac{n(\text{no flu and treatment})}{n(\text{total patients})} \\ &= \frac{252}{552} = \frac{21}{46} \end{aligned}$$

e) Using a calculation, determine whether a patient receiving the treatment and having no flu after three days are dependent or independent events.

Solution:

$$\begin{aligned} P(\text{treatment}) \times P(\text{no flu}) &= \frac{11}{23} \times \frac{13}{23} = \frac{143}{529} = 0{,}270 \\ P(\text{treatment and no flu} &= \frac{21}{46} = 0{,}457 \end{aligned}$$

Therefore receiving treatment and having no flu after three days are dependent events.

f) Calculate the probability that a patient receiving treatment will have no flu after three days.

Solution:

$$\begin{aligned} P(\text{no flu if treated}) &= \frac{n(\text{no flu and treatment})}{n(\text{total treated})} \\ &= \frac{252}{264} = \frac{21}{22} \end{aligned}$$

g) Calculate the probability that a patient receiving a placebo will have no flu after three days.

Solution:

$$\begin{aligned} P(\text{no flu if given placebo}) &= \frac{n(\text{no flu and placebo})}{n(\text{total placebo})} \\ &= \frac{60}{288} = \frac{5}{24} \end{aligned}$$

h) Comparing you answers in f) and g), would you recommend the use of the new treatment for patients suffering from influenza?

Solution:

The probability of having no influenza after three days is much higher when on the new treatment so its use is recommended.

i) A hospital is trying to decide whether to purchase the new treatment. The new treatment is much more expensive than the old treatment. According to the hospital records, of the 72 024 flu patients that have been treated with the old treatment, only 3200 still had the flu three days after treatment.

- Construct a two-way contingency table comparing the old treatment data with the new treatment data.
- Using the data from your table, advise the hospital whether to purchase the new treatment or not.

Solution:

	Flu	**No flu**	**Total**
Old treatment	3200	68 824	72 024
New treatment	12	252	264
Total	3212	69 076	72 288

$$\begin{aligned} P(\text{no flu if old treatment}) &= \frac{68\,824}{72\,024} \\ &= \frac{8603}{9003} = 0{,}956 \\ P(\text{no flu if new treatment}) &= \frac{252}{264} == 0{,}955 \end{aligned}$$

The probability of not having flu after three days if given the new treatment is approximately the same if given the old treatment, therefore the hospital should not purchase the new, more expensive treatment.

5. Human immunodeficiency virus (HIV) affects 10% of the South African population.

a) If a test for HIV has a 99,9% accuracy rate (i.e. 99,9% of the time the test is correct, 0,1% of the time, the test returns a false result), draw a two-way contingency table showing the expected results if 10 000 of the general population are tested.

Solution:

If 10 000 people are tested and the prevalence rate is 10%:

$$\begin{aligned} 10\ 000 \times 0{,}1 &= 1000 \text{ people are expected to be sick} \\ \text{Therefore } 10\ 000 - 1000 &= 9000 \text{ people are expected to be healthy} \end{aligned}$$

	Sick	**Healthy**	**Total**
Positive			
Negative			
Total	1000	9000	10 000

If the test is 99,9% accurate:

$$\begin{aligned} 1000 \times 0{,}999 &= 999 \text{ sick people are expected to test positive} \\ \text{Therefore } 1000 - 999 &= 1 \text{ sick person is expected to test negative} \\ \text{And } 9000 \times 0{,}999 &= 8991 \text{ healthy people are expected to test negative} \\ \text{Therefore } 9000 - 8991 &= 9 \text{ healthy people are expected to test positive} \end{aligned}$$

	Sick	**Healthy**	**Total**
Positive	999	9	1008
Negative	1	8991	8992
Total	1000	9000	10 000

b) Calculate the probability that a person who tests positive for HIV does not have the disease, correct to two decimal places.

Solution:

$$\begin{aligned} P(\text{healthy if tested positive}) &= \frac{n(\text{healthy and tested positive})}{n(\text{tested positive})} \\ &= \frac{9}{1008} \\ &= 0{,}01 \end{aligned}$$

It is worth noting that this probability is bigger than the one suggested by the '99,9% accuracy' of the test.

c) In practice, a person who tests positive for HIV is always tested a second time. Calculate the probability that an HIV-negative person will test positive after two tests, correct to four decimal places.

Solution:

$$\begin{aligned} P(\text{healthy if tested positive twice}) &= \frac{9}{1008} \times \frac{9}{1008} \\ &= \frac{81}{1\ 016\ 064} \\ &= 0{,}0001 \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29GN 2. 29GP 3. 29GQ 4. 29GR 5. 29GS

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Exercise 10 – 4: Number of possible outcomes if repetition is allowed

1. Tarryn has five different skirts, four different tops and three pairs of shoes. Assuming that all the colours complement each other, how many different outfits can she put together?

   Solution:

   $$5 \times 4 \times 3 = 60 \text{ different outfits}$$

2. In a multiple-choice question paper of 20 questions the answers can be A, B, C or D. How many different ways are there of answering the question paper?

   Solution:

   $$4^{20} = 1{,}0995 \times 10^{12} \text{ different ways of answering the exam paper}$$

3. A debit card requires a five digit personal identification number (PIN) consisting of digits from 0 to 9. The digits may be repeated. How many possible PINs are there?

   Solution:

   $$10^5 = 100\ 000 \text{ possible PINs}$$

4. The province of Gauteng ran out of unique number plates in 2010. Prior to 2010, the number plates were formulated using the style LLLDDDGP, where L is any letter of the alphabet excluding vowels and Q, and D is a digit between 0 and 9. The new style the Gauteng government introduced is LLDDLLGP. How many more possible number plates are there using the new style when compared to the old style?

   Solution:

   $$\text{Old style: } 20^3 \times 10^3 = 8\ 000\ 000 \text{ possible arrangements}$$
   $$\text{New style: } 20^4 \times 10^2 = 16\ 000\ 000 \text{ possible arrangements}$$
   $$16\ 000\ 000 - 8\ 000\ 000 = 8\ 000\ 000$$

   Therefore there are $8\ 000\ 000$ more possible number plates using the new style.

5. A gift basket is made up from one CD, one book, one box of sweets, one packet of nuts and one bottle of fruit juice. The person who makes up the gift basket can choose from five different CDs, eight different books, three different boxes of sweets, four kinds of nuts and six flavours of fruit juice. How many different gift baskets can be produced?

   Solution:

   $$5 \times 8 \times 3 \times 4 \times 6 = 2880 \text{ possible gift baskets}$$

6. The code for a safe is of the form XXXXYYY where X is any number from 0 to 9 and Y represents the letters of the alphabet. How many codes are possible for each of the following cases:

   a) the digits and letters of the alphabet can be repeated.

      Solution:

      $$10^4 \times 26^3 = 175\ 760\ 000 \text{ possible codes}$$

   b) the digits and letters of the alphabet can be repeated, but the code may not contain a zero or any of the vowels in the alphabet.

      Solution:

      We exclude the digit 0 and the vowels (A; E; I; O; U), leaving 9 other digits and 21 letters to choose from.

      $$9^4 \times 21^3 = 60\ 761\ 421 \text{ possible codes}$$

c) the digits and letters of the alphabet can be repeated, but the digits may only be prime numbers and the letters X, Y and Z are excluded from the code.

Solution:

The prime digits are 2, 3, 5 and 7. This gives us 4 possible digits. If we exclude the letters X, Y and Z, we are left with 23 letters to choose from.

$$4^4 \times 23^3 = 3\ 114\ 752 \text{ possible codes}$$

7. A restaurant offers four choices of starter, eight choices for the main meal and six choices for dessert. A customer can choose to eat just one course, two different courses or all three courses. Assuming that all courses are available, how many different meal options does the restaurant offer?

Solution:

- A person who eats only a starter has 4 choices
- A person who eats only a main meal has 8 choices
- A person who eats only a dessert has 6 choices
- A person who eats a starter and a main course has $4 \times 8 = 32$ choices
- A person who eats a starter and a dessert has $4 \times 6 = 24$ choices
- A person who eats a main meal and a dessert has $8 \times 6 = 48$ choices
- A person who eats all three courses has $4 \times 8 \times 6 = 192$ choices.

Therefore, there are $4 + 8 + 6 + 32 + 24 + 48 + 192 = 314$ different meal options

Check answers online with the exercise code below or click on 'show me the answer'.
1. 29GV 2. 29GW 3. 29GX 4. 29GY 5. 29GZ 6. 29H2
7. 29H3

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10.5 Factorial notation

Exercise 10 – 5: Factorial notation

1. Work out the following without using a calculator:

a) 3!

Solution:

$$3 \times 2 \times 1 = 6$$

b) 6!

Solution:

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

c) 2!3!

Solution:

$$2 \times 1 \times 3 \times 2 \times 1 = 12$$

d) 8!

Solution:

$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40\ 320$$

e) $\frac{6!}{3!}$

Solution:

$$\frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 120$$

f) $6! + 4! - 3!$

Solution:

$$(6 \times 5 \times 4 \times 3 \times 2 \times 1) + (4 \times 3 \times 2 \times 1) - (3 \times 2 \times 1) = 720 + 24 - 6 = 738$$

g) $\frac{6! - 2!}{2!}$

Solution:

$$\frac{(6 \times 5 \times 4 \times 3 \times 2 \times 1) - (2 \times 1)}{2 \times 1} = \frac{720 - 2}{2} = 359$$

h) $\frac{2! + 3!}{5!}$

Solution:

$$\frac{(2 \times 1) + (3 \times 2 \times 1)}{5 \times 4 \times 3 \times 2 \times 1} = \frac{2 + 6}{120} = \frac{1}{15}$$

i) $\frac{2! + 3! - 5!}{3! - 2!}$

Solution:

$$\frac{2 + 6 - 120}{6 - 2} = \frac{-112}{4} = -28$$

j) $(3!)^3$

Solution:

$6 \times 6 \times 6 = 216$

k) $\frac{3! \times 4!}{2!}$

Solution:

$$\frac{(3 \times 2 \times 1) \times (4 \times 3 \times 2 \times 1)}{2 \times 1} = 72$$

2. Calculate the following using a calculator:

a) $\frac{12!}{2!}$

Solution:

$$\frac{(12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)}{2 \times 1} = 239\ 500\ 800$$

b) $\frac{10!}{20!}$

Solution:

$1{,}49 \times 10^{-12}$

c) $\frac{10! + 12!}{5! + 6!}$

Solution:

574 560

d) $5!(2! + 3!)$

Solution:

960

e) $(4!)^2(3!)^2$

Solution:

20 736

3. Show that the following is true:

a) $\frac{n!}{(n-2)!} = n^2 - n$

Solution:

$$\frac{n!}{(n-2)!} = \frac{n \times (n-1) \times \cancel{(n-2)} \times \cancel{(n-3)} \times \ldots \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{(n-2)} \times \cancel{(n-3)} \times \ldots \times \cancel{3} \times \cancel{2} \times \cancel{1}}$$

$$= n(n-1) = n^2 - n$$

b) $\frac{(n-1)!}{n!} = \frac{1}{n}$

Solution:

$$\frac{(n-1)!}{n!} = \frac{\cancel{(n-1)} \times \cancel{(n-2)} \times \cancel{(n-3)} \times \ldots \times \cancel{3} \times \cancel{2} \times \cancel{1}}{n \times \cancel{(n-1)} \times \cancel{(n-2)} \times \cancel{(n-3)} \times \ldots \times \cancel{3} \times \cancel{2} \times \cancel{1}} = \frac{1}{n}$$

c) $\frac{(n-2)!}{(n-1)!} = \frac{1}{n-1}$ for $n > 1$

Solution:

$$\frac{(n-2)!}{(n-1)!} = \frac{\cancel{(n-2)} \times \cancel{(n-3)} \times \ldots \times \cancel{3} \times \cancel{2} \times \cancel{1}}{(n-1) \times \cancel{(n-2)} \times \cancel{(n-3)} \times \ldots \times \cancel{3} \times \cancel{2} \times \cancel{1}} = \frac{1}{n-1}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 29H4	1b. 29H5	1c. 29H6	1d. 29H7	1e. 29H8	1f. 29H9
1g. 29HB	1h. 29HC	1i. 29HD	1j. 29HF	1k. 29HG	2a. 29HH
2b. 29HJ	2c. 29HK	2d. 29HM	2e. 29HN	3a. 29HP	3b. 29HQ
3c. 29HR					

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10.6 Application to counting problems

Exercise 10 – 6: Number of choices in a row

1. How many different possible outcomes are there for a swimming event with six competitors?

   Solution:

   $6! = 720$

2. How many different possible outcomes are there for the gold (1st), silver (2nd) and bronze (3rd) medals in a swimming event with six competitors?

   Solution:

   $6 \times 5 \times 4 = 120$

3. Susan wants to visit her friends in Pretoria, Johannesburg, Phalaborwa, East London and Port Elizabeth. In how many different ways can the visits be arranged?

   Solution:

   $5! = 120$ ways

4. A head boy, a deputy head boy, a head girl and a deputy head girl must be chosen out of a student council consisting of 18 girls and 18 boys. In how many ways can they be chosen?

Solution:

$18 \times 17 + 18 \times 17 = 612$ ways

5. Twenty different people enter a golf competition. Only the first six of them can win prizes. In how many different ways can the prizes be won?

Solution:

$20 \times 19 \times 18 \times 17 \times 16 \times 15 = 27\ 907\ 200$ ways

6. Three letters of the word 'EMPTY' are arranged in a row. How many different arrangements are possible?

Solution:

$5 \times 4 \times 3 = 60$ arrangements

7. Pool balls are numbered from 1 to 15. You have only one set of pool balls. In how many different ways can you arrange:

   a) all 15 balls. Write your answer in scientific notation, rounding off to two decimal places.

   Solution:

   $15! = 1{,}31 \times 10^{12}$

   b) four of the 15 balls.

   Solution:

   $15 \times 14 \times 13 \times 12 = 32\ 760$

8. The captains of all the sports teams in a school have to stand next to each other for a photograph. The school sports programme offers rugby, cricket, hockey, soccer, netball and tennis.

   a) In how many different orders can they stand in the photograph?

   Solution:

   $$6! = 720 \text{ different orders}$$

   b) In how many different orders can they stand in the photograph if the rugby captain stands on the extreme left and the cricket captain stands on the extreme right?

   Solution:

   Since we have no choice about where to put the rugby and cricket captains, there are only 4 people left to arrange.

   $$4! = 24 \text{ different orders}$$

   c) In how many different orders can they stand if the rugby captain, netball captain and cricket captain must stand next to each other?

   Solution:

   The rugby captain, netball captain and cricket captain are treated as a single object, as they must stand together. So there are four different objects to arrange therefore there are 4! different arrangements. The rugby captain, netball captain and cricket captain can also swop positions between themselves in 3! different ways, therefore there are:

   $$4! \times 3! = 144 \text{ different orders}$$

9. How many three-digit numbers can be made from the digits 1 to 6 if:

   a) repetition is not allowed?

   Solution:

   $$6 \times 5 \times 4 = 120$$

   b) repetition is allowed?

Solution:

$$6^3 = 216$$

10. There are two different red books and three different blue books on a shelf.

    a) In how many different ways can these books be arranged?
    Solution:

    $$5! = 120 \text{ different ways to arrange the books}$$

    b) If you want the red books to be together, in how many different ways can the books be arranged?
    Solution:
    If the red books are treated as a single object, there are four different objects to arrange therefore there are $4!$ different arrangements. The red books can also be rearranged between themselves in $2!$ different ways, therefore there are:

    $$4! \times 2! = 48 \text{ different ways to arrange the books}$$

    c) If you want all the red books to be together and all the blue books to be together, in how many different ways can the books be arranged?
    Solution:
    There are two groups of books, red and blue, which can be arranged $2!$ ways. Then there are two red books which can be arranged in $2!$ ways and there are three blue books which can be arranged in $3!$ ways. Therefore, there are

    $$2! \times 2! \times 3! = 24 \text{ different ways to arrange the books}$$

11. There are two different Mathematics books, three different Natural Sciences books, two different Life Sciences books and four different Accounting books on a shelf. In how many different ways can they be arranged if:

    a) the order does not matter?
    Solution:

    $$11! = 39\ 916\ 800 \text{ ways to arrange the books}$$

    b) all the books of the same subject stand together?
    Solution:
    There are four groups of books, which can be arranged in $4!$ different ways. Of the books, two are Mathematics books, three are Natural Sciences books, two are Life Sciences books and four are Accounting books. Therefore, there are:

    $$4! \times 2! \times 3! \times 2! \times 4! = 13\ 824 \text{ ways to arrange the books}$$

    c) the two Mathematics books stand first?
    Solution:
    The Mathematics books can be arranged in $2!$ ways while the remaining books can be arranged in $9!$ ways. Therefore, there are:

    $$2! \times 9! = 725\ 760 \text{ ways to arrange the books}$$

    d) the Accounting books stand next to each other?
    Solution:
    The Accounting books can be arranged in $4!$ ways and, if treated as a single object, can be arranged with the remaining books in $8!$ ways. Therefore, there are:

    $$4! \times 8! = 967\ 680$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29HS 2. 29HT 3. 29HV 4. 29HW 5. 29HX 6. 29HY
7. 29HZ 8. 29J2 9. 29J3 10. 29J4 11. 29J5

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Exercise 10 – 7: Number of arrangements of sets containing alike objects

1. You have the word 'EXCELLENT'.

   a) If the repeated letters are regarded as different letters, how many letter arrangements are possible?

   Solution:

   $$9! = 362\ 880$$

   b) If the repeated letters are regarded as identical, how many letter arrangements are possible?

   Solution:

   There are 3 E's and 2 L's so we have to divide by $3!$ and $2!$.

   $$\frac{9!}{3! \times 2!} = 30\ 240$$

   c) If the first and last letters are identical, how many letter arrangements are there?

   Solution:

   The word could start and end in E or L. With an E, there are $\frac{7!}{2!}$ letter arrangements (divide by 2 L's) and with an L there are $\frac{7!}{3!}$ letter arrangements (divide by 3 E's). Therefore, there are:

   $$\frac{7!}{3!} + \frac{7!}{2!} = 840 + 2520 = 3360 \text{ possible letter arrangements}$$

   d) How many letter arrangements can be made if the arrangement starts with an L?

   Solution:

   This is equivalent to removing one L from the letters available for arrangement. Therefore, there are:

   $$\frac{8!}{3!} = 6720 \text{ possible letter arrangements}$$

   e) How many letter arrangements are possible if the word ends in a T?

   Solution:

   This is equivalent to removing the T from the letters available for arrangement. Therefore, there are:

   $$\frac{8!}{3! \times 2!} = 3360 \text{ possible letter arrangements}$$

2. You have the word 'ASSESSMENT'.

   a) If the repeated letters are regarded as different letters, how many letter arrangements are possible?

   Solution:

   $$10! = 3\ 628\ 800$$

   b) If the repeated letters are regarded as identical, how many letter arrangements are possible?

   Solution:

   $$\frac{10!}{4! \times 2!} = 75\ 600$$

   c) If the first and last letters are identical, how many letter arrangements are there?

   Solution:

   The word could start and end in S or E. With an S, there are $\frac{8!}{2! \times 2!}$ letter arrangements (divide by 2 E's and 2 remaining S's) and with an E there are $\frac{8!}{4!}$ letter arrangements (divide by 4 S's). Therefore, there are:

   $$\frac{8!}{2! \times 2!} + \frac{8!}{4!} = 10\ 080\ + 1680 = 11\ 760 \text{ possible letter arrangements}$$

d) How many letter arrangements can be made if the arrangement starts with a vowel?

Solution:

The word could start with A or E. With an A, there are $\frac{9!}{4! \times 2!}$ letter arrangements (divide by 2 E's and 4 S's) and with an E there are $\frac{9!}{4!}$ letter arrangements (divide by 4 S's). Therefore, there are:

$$\frac{9!}{4! \times 2!} + \frac{9!}{4!} = 7560 + 15\,120 = 22\,680 \text{ possible letter arrangements}$$

e) How many letter arrangements are possible if all the S's are at the beginning of the word?

Solution:

This is equivalent to removing all the S's from the letters available for arrangement. Therefore, there are:

$$\frac{6!}{2!} = 360 \text{ possible letter arrangements}$$

3. On a piano the white keys represent the following notes: C, D, E, F, G, A, B. How many tunes, seven notes in length, can be composed with these notes if:

a) a note can be played only once?

Solution:

$$7! = 5040 \text{ possible tunes}$$

b) the notes can be repeated?

Solution:

$$7^7 = 823\,543 \text{ possible tunes}$$

c) the notes can be repeated and the tune begins and ends with a D?

Solution:

The tune starting and ending with a D leaves five possible positions in which to arrange the seven notes. Therefore, there are:

$$7^5 = 16\,807 \text{ possible tunes}$$

d) the tune consists of 3 D's, 2 B's and 2 A's.

Solution:

$$\frac{7!}{3! \times 2! \times 2!} = 210 \text{ possible tunes}$$

4. There are three black beads and four white beads in a row. In how many ways can the beads be arranged if:

a) same-coloured beads are treated as different beads?

Solution:

$$7! = 5040 \text{ ways}$$

b) same-coloured beads are treated as identical beads?

Solution:

$$\frac{7!}{3! \times 4!} = 35 \text{ ways}$$

5. There are eight balls on a table. Some are white and some are red. The white balls are all identical and the red balls are all identical. The balls are removed one at a time. In how many different orders can the balls be removed if:

a) seven of the balls are red?

Solution:

$$\frac{8!}{7!} = 8 \text{ different orders}$$

b) three of the balls are red?

Solution:

$$\frac{8!}{3! \times 5!} = 56 \text{ different orders}$$

c) there are four of each colour?

Solution:

$$\frac{8!}{4! \times 4!} = 70 \text{ different orders}$$

6. How many four-digit numbers can be formed with the digits 3, 4, 6 and 7 if:

a) there can be repetition?

Solution:

$$4^4 = 256 \text{ possible numbers}$$

b) each digit can only be used once?

Solution:

$$4! = 24 \text{ possible numbers}$$

c) if the number is odd and repetition is allowed?

Solution:

For the number to be odd, it must end in 3 or 7. For the numbers ending in 3, there are 4^3 different arrangements of the first three digits and similarly, for 7, there are 4^3 different arrangements. Therefore, there are

$$4^3 + 4^3 = 128 \text{ possible numbers which match the criteria}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29J6 2. 29J7 3. 29J8 4. 29J9 5. 29JB 6. 29JC

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10.7 Application to probability problems

Exercise 10 – 8: Solving probability problems using the fundamental counting principle

1. A music group plans a concert tour in South Africa. They will perform in Cape Town, Port Elizabeth, Pretoria, Johannesburg, Bloemfontein, Durban and East London.

a) In how many different orders can they plan their tour if there are no restrictions?

Solution:

$$7! = 5040 \text{ different orders are possible}$$

b) In how many different orders can they plan their tour if their tour begins in Cape Town and ends in Durban?

Solution:

This reduces the available objects (cities) by two, hence

$$5! = 120 \text{ different orders are possible}$$

c) If the tour cities are chosen at random, what is the probability that their performances in Cape Town, Port Elizabeth, Durban and East London happen consecutively? Give your answer correct to 3 decimal places.

Solution:

If these cities are grouped together, they can be treated as a single object in the arrangement, hence there are $4!$ different ways to order the objects. Within the grouped cities, there are $4!$ different ways to order them. Therefore, there are $4! \times 4! = 576$ different orders

Therefore, the probability of any of the orders having Cape Town, Port Elizabeth, Durban and East London happen consecutively is:

$$\begin{aligned} P(\text{the 4 cities grouped}) &= \frac{n(\text{the 4 cities grouped})}{n(\text{total possible orders})} \\ &= \frac{576}{5040} = 0{,}114 \end{aligned}$$

2. A certain restaurant has the following course options available for a three-course set menu:

STARTERS	MAINS	DESSERTS
Calamari salad	Fried chicken	Ice cream and chocolate sauce
Oysters	Crumbed lamb chops	Strawberries and cream
Fish in garlic sauce	Mutton Bobotie	Malva pudding with custard
	Chicken schnitzel	Pears in brandy sauce
	Vegetable lasagne	
	Chicken nuggets	

a) How many different set menus are possible?

Solution:

$$3 \times 6 \times 4 = 72 \text{ different set menus}$$

b) What is the probability that a set menu includes a chicken course?

Solution:

$$\begin{aligned} n(\text{set menu with chicken}) &= 3 \times 3 \times 4 = 36 \\ n(\text{total set menus}) &= 72 \\ \text{Therefore } P(\text{set menu with chicken}) &= \frac{36}{72} = 0{,}5 \end{aligned}$$

3. Eight different pairs of jeans and 5 different shirts hang on a rail.

a) In how many different ways can the clothes be arranged on the rail?

Solution:

$$13! = 6\ 227\ 020\ 800 \text{ different ways}$$

b) In how many ways can the clothing be arranged if all the jeans hang together and all the shirts hang together?

Solution:

The shirts and jeans form two groups, which can be arranged $2!$ ways. The five shirts can be arranged $5!$ ways and the eight pairs of jeans can be arranged $8!$ ways.

$$2! \times 8! \times 5! = 9\ 676\ 800$$

c) What is the probability, correct to three decimal places, of the clothing being arranged on the rail with a shirt at one end and a pair of jeans at the other?

Solution:

- The five different choices of shirt and eight different choices of pairs of jeans can form 5×8 different arrangements at the ends of the rail.
- There are $2!$ different ways to arrange a shirt at one end and a pair of jeans on the other: S - - - - - - - - - - - J and J - - - - - - - - - - - S

- If a shirt is at one end and a pair of jeans at the other, there remains $11!$ different arrangements of the remaining clothing items.

Therefore, there are:

$2 \times 8 \times 5 \times 11! = 3\ 193\ 344\ 000$ different ways to arrange the clothing

The probability of a clothing arrangement with a shirt at one end and a pair of jeans at the other $= \frac{3\ 193\ 344\ 000}{6\ 227\ 020\ 800} = 0{,}513$

4. A photographer places eight chairs in a row in his studio in order to take a photograph of the debating team. The team consists of three boys and five girls.

   a) In how many ways can the debating team be seated?

   Solution:

   $$8! = 40\ 320$$

   b) What is the probability that a particular boy and a particular girl sit next to each other?

   Solution:

   Regard the particular boy and girl as one group. The group can be seated $2!$ ways. The number of ways that this group and the remaining 6 people can sit $= 7!$. Therefore the total number of ways this particular boy and girl can sit together in the photograph $= 2! \times 7! = 10\ 080$. The probability of a particular boy and girl sitting together $= \frac{10\ 080}{40\ 320} = 0{,}25$

5. If the letters of the word 'COMMITTEE' are randomly arranged, what is the probability that the letter arrangements start and end with the same letter?

   Solution:

   - There are 2 M's, 2 T's and 2 E's and a total of 9 letters.
   - Total number of letter arrangements $= \dfrac{9!}{2! \times 2! \times 2!} = 45\ 360$
   - Possibilities of the first and last letter being the same:
     - M(COITTEE)M
       Total of 7 letters of which there are 2E's and 2T's
       Number of letter arrangements $= \dfrac{7!}{2! \times 2!} = 1260$
     - T(MCOIEEM)T
       Total of 7 letters of which there are 2E's and 2M's
       Number of letter arrangements $= \dfrac{7!}{2! \times 2!} = 1260$
     - E(TTMCOIM)E
       Total of 7 letters of which there are 2T's and 2M's
       Number of letter arrangements $= \dfrac{7!}{2! \times 2!} = 1260$
   - Total number of letter arrangements if the letter arrangement starts and ends with the same letter $= 3 \times 1260 = 3780$.

   $$P(\text{first and last letter the same}) = \frac{3780}{45\ 360} = \frac{1}{12}$$

6. Four different Mathematics books, three different Economics books and two different Geography books are arranged on a shelf. What is the probability that all the books of the same subject are arranged next to each other?

   Solution:

   Total number of different ways the books can be arranged $= 9! = 362\ 880$.There are 3 subjects of books which can be arranged $3!$ ways.

   Therefore, the total number of arrangements if the subjects are arranged together $= 3! \times 4! \times 3! \times 2! = 1728$

   $$P(\text{books of the same subject next to each other}) = \frac{1728}{362\ 880} = \frac{1}{210}$$

7. A number plate is made up of three letters of the alphabet (excluding F and S) followed by three digits from 0 to 9. The numbers and letters can be repeated. Calculate the probability that a randomly chosen number plate:

a) starts with the letter D and ends with the digit 3.

Solution:

Total number of arrangements $= 24^3 \times 10^3$

Total number of arrangements beginning with D and ending with 3 $= 24^2 \times 10^2$

$$P(\text{first D and last } 3) = \frac{24^2 \times 10^2}{24^3 \times 10^3} = \frac{1}{240}$$

b) has precisely one D.

Solution:

The 'D' could be at the first, second or third positions. The other two letters cannot include a 'D', leaving 23 other letters. Therefore there are

$23^2 \times 10^3 \times 3$ different arrangements containing only one D

$$\text{Therefore } P(\text{only one D}) = \frac{23^2 \times 10^3 \times 3}{24^3 \times 10^3} = \frac{529}{4608}$$

c) contains at least one 5.

Solution:

$$\begin{aligned} P(\text{contains at least one } 5) &= 1 - P(\text{no 5s}) \\ &= 1 - \frac{24^3 \times 9^3}{24^3 \times 10^3} \\ &= 1 - 0{,}729 = 0{,}271 \end{aligned}$$

8. In the 13-digit identification (ID) numbers of South African citizens:

- The first six numbers are the birth date of the person in YYMMDD format.
- The next four digits indicate gender, with 5000 and above being male and 0001 to 4999 being female.
- The next number is the country ID; 0 is South Africa and 1 is not.
- The second last number used to be a racial identifier but it is now 8 for everybody.
- The last number is a control digit, which verifies the rest of the number.

Assume that the control digit is a randomly generated digit from 0 to 9 and ignore the fact that leap years have an extra day.

a) Calculate the total number of possible ID numbers.

Solution:

For all available arrangements, if an ID number is structured ABCDEFGHIJKLM:

- A and B are any digits between 00 and 99 (year)
- C and D are any digits from 01 to 12 (month)
- E and F are any digits from 01 to 28, 30 or 31 dependent on month (day)
- G, H, I and J are any digits from 0001 to 9999 (gender)
- K is either a 0 or 1
- L is an 8
- M is any digit between 0 and 9

Therefore, the total number of possible ID numbers for 30-day months is:

$$100 \times 4 \times 30 \times 9999 \times 2 \times 1 \times 10 = 2\ 399\ 760\ 000$$

Therefore, the total number of possible ID numbers for 31-day months is:

$$100 \times 7 \times 31 \times 9999 \times 2 \times 1 \times 10 = 4\ 339\ 566\ 000$$

Therefore, the total number of possible ID numbers for February is:

$$100 \times 1 \times 28 \times 9999 \times 2 \times 1 \times 10 = 559\ 944\ 000$$

Therefore, the total number of possible ID numbers is:

$$2\ 399\ 760\ 000 + 4\ 339\ 566\ 000 + 559\ 944\ 000 = 9\ 239\ 076\ 000$$

b) Calculate the probability that a randomly generated ID number is of a South African male born during the 1980s. Write your answer correct to two decimal places.

Solution:

There is a lot of information in the problem and we can simplify it by identifying the relevant information. We want to calculate the probability that an ID number is for a South African male born during the 1980s. This means that we have to look at the digits for country, gender and year of birth.

If an ID number is structured ABCDEFGHIJKLM:

- For a South African, K is 0.
- For a male, GHIJ are from 5000 to 9999.
- For someone born during the 1980s, AB are between 80 and 89.

This gives $1 \times 5000 \times 10 = 50\ 000$ combinations for ABGHIJK

Without any restrictions, the total combinations for ABGHIJK is $2 \times 9999 \times 100 = 1\ 999\ 800$.

$$\text{Therefore } P(\text{SA male 80s}) = \frac{50\ 000}{1\ 999\ 800} = 0{,}025$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29JD 2. 29JF 3. 29JG 4. 29JH 5. 29JJ 6. 29JK
7. 29JM 8. 29JN

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10.8 Summary

Exercise 10 – 9: End of chapter exercises

1. An ATM card has a four-digit PIN. The four digits can be repeated and each of them can be chosen from the digits 0 to 9.

   a) What is the total number of possible PINs?

   Solution:

   $$10^4 = 10\ 000$$

   b) What is the probability of guessing the first digit correctly?

   Solution:

   $\frac{1}{10}$

   c) What is the probability of guessing the second digit correctly?

   Solution:

   $\frac{1}{10}$

   d) If your ATM card is stolen, what is the probability, correct to four decimal places, of a thief guessing all four digits correctly on his first guess?

   Solution:

   $\left(\frac{1}{10}\right)^4 = 0{,}0001$

   e) After three incorrect PIN attempts, an ATM card is blocked from being used. If your ATM card is stolen, what is the probability, correct to four decimal places, of a thief blocking the card? Assume the thief enters a different PIN each time.

   Solution:

$$P(\text{incorrect PIN}) = 1 - P(\text{correct PIN})$$
$$= 1 - \frac{1}{10^4} = 0{,}9999$$
$$\text{Therefore } P(3 \text{ incorrect PIN attempts}) = (0{,}9999)^3$$
$$= 0{,}9997$$

2. The LOTTO rules state the following:

   - Six numbers are drawn from the numbers 1 to 49 - this is called a 'draw'.
   - Numbers are not replaced once drawn, so you cannot have the same number more than once.
   - The order of the drawn numbers does not matter.

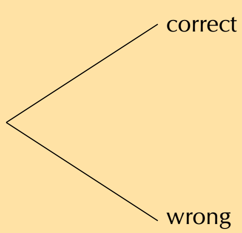
   You decide to buy one LOTTO ticket consisting of 6 numbers.

   a) How many different possible LOTTO draws are there? Write your answer in scientific notation, rounding to two digits after the decimal point.

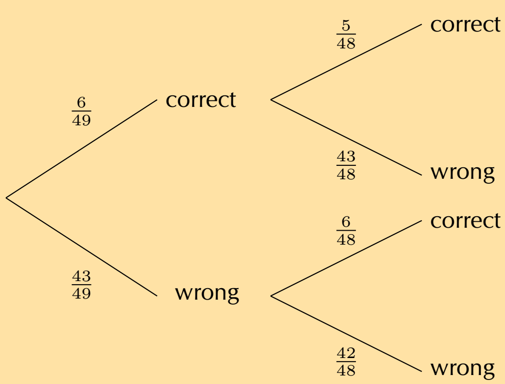
   Solution:

   $$49 \times 48 \times 47 \times 46 \times 45 \times 44 = 1{,}01 \times 10^{10}$$

   b) Complete the tree diagram below after the first two LOTTO numbers have been drawn showing the possible outcomes and probabilities of the numbers on your ticket.



   Solution:



   c) What is the probability of getting the first number drawn correctly?

   Solution:

   $$P(\text{first number correct}) = \frac{6}{49}$$

   d) What is the probability of getting the second number drawn correctly if you get the first number correct?

   Solution:

   $$P(\text{second number correct if first correct}) = \frac{5}{48}$$

   e) What is the probability of getting the second number drawn correct if you do not get the first number correctly?

Solution:

$$P(\text{second number correct if first incorrect}) = \frac{6}{48}$$

f) What is the probability of getting the second number drawn correct?

Solution:

This is the sum of the probabilities of the outcomes where the second number drawn is correct:

$$\begin{aligned} P(\text{second number correct}) &= \frac{6}{49} \times \frac{5}{48} + \frac{43}{49} \times \frac{6}{48} \\ &= \frac{5}{392} + \frac{43}{392} \\ &= \frac{6}{49} \end{aligned}$$

Notice that the answer is the same as the probability of getting the first number correct. If you are unaware of the outcome of prior events, the probability of the outcome of a certain event, is equal to the probability of that outcome of the first event. Learners are not required to learn this concept but it is interesting to note.

g) What is the probability of getting all 6 LOTTO numbers correct? Write your answer in scientific notation, rounding to two digits after the decimal point.

Solution:

$$P(\text{all 6 correct}) = \frac{6}{49} \times \frac{5}{48} \times \frac{4}{47} \times \frac{3}{46} \times \frac{2}{45} \times \frac{1}{44} = 7{,}15 \times 10^{-8}$$

3. The population statistics of South Africa show that 55% of all babies born are female. Calculate the probability that a couple planning to have children will have a boy followed by a girl and then a boy. Assume that each birth is an independent event. Write your answer as a percentage, correct to two decimal places.

   Solution:

   $$0{,}45 \times 0{,}55 \times 0{,}45 = 11{,}14\%$$

4. Fezile and Vuzi write a Mathematics test. The probability that Fezile will pass the test is $0{,}8$. The probability that Vuzi will pass the test is $0{,}75$. What is the probability that only one of them will pass the test?

   Solution:

   $$\begin{aligned} P(\text{only one passes}) &= P(\text{F pass}) \times P(\text{V fail}) + P(\text{F fail}) \times P(\text{V pass}) \\ &= 0{,}8 \times 0{,}25 + 0{,}2 \times 0{,}75 \\ &= 0{,}35 \end{aligned}$$

5. Landline telephone numbers are 10 digits long. Numbers begin with a zero followed by 9 digits chosen from the digits 0 to 9. Repetitions are allowed.

   a) How many different phone numbers are possible?

   Solution:

   $$10^9 = 1\ 000\ 000\ 000$$

   b) The first three digits of a number form an area code. The area code for Cape Town is 021. How many different phone numbers are available in the Cape Town area?

   Solution:

   $$10^7 = 10\ 000\ 000$$

   c) What is the probability of the second digit being an even number?

Solution:

There are 5 even numbers between 0 and 9, therefore:

$$P(\text{second digit even}) = \frac{5}{10} = \frac{1}{2}$$

d) Ignoring the first digit, what is the probability of a phone number consisting of only odd digits? Write your answer correct to three decimal places.

Solution:

$$\left(\frac{5}{10}\right)^9 = 0{,}002$$

6. Take the word 'POSSIBILITY'.

a) In how many way can the letters be arranged if repeated letters are considered identical?

Solution:

There are two S's and three I's, therefore there are:

$$\frac{11!}{2! \times 3!} = 3\ 326\ 400 \text{ different arrangements}$$

b) What is the probability that a randomly generated arrangement of the letters will begin with three I's? Write your answer as a fraction.

Solution:

If the arrangement begins with three I's, that leaves POSSBLTY to be arranged.

$$\begin{aligned} n(\text{arrangements beginning with III}) &= \frac{8!}{2!} = 20\ 160 \\ \text{Therefore } P(\text{arrangements beginning with III}) &= \frac{20\ 160}{3\ 326\ 400} \\ &= \frac{1}{165} \end{aligned}$$

7. The code to a safe consists of 10 digits chosen from the digits 0 to 9. None of the digits are repeated. Determine the probability of a code where the first digit is odd and none of the first three digits may be a zero. Write your answer as a percentage, correct to two decimal places.

Solution:

- There are 5 odd numbers from 0 to 9, so there are five arrangements for the first digit.
- For the second digit there are 8 digits left to arrange, as the first digit and zero are removed.
- For the third digit there are 7 digits left to arrange as zero and the first two digits are removed.
- For the fourth digit, there are 7 digits (0 is now included) left which can be combined in 7! ways for the remaining digits.

$$\begin{aligned} \text{Total possible number of codes} &= 10! = 3\ 628\ 800 \\ \text{Total no. of codes with first digit odd, first three non-zero} &= 5 \times 8 \times 7 \times 7! \\ &= 1\ 411\ 200 \\ \text{Therefore } P(\text{first digit odd, first three non-zero}) &= \frac{1\ 411\ 200}{3\ 628\ 800} \\ &= 38{,}89\% \end{aligned}$$

8. Four different red books and three different blue books are to be arranged on a shelf. What is the probability that all the red books and all the blue books stand together on the shelf?

Solution:

Total number of arrangements $= 7!$

If the red and blue books stand together, there are two groups of books which can be arranged $2!$ ways.

Then there are four red books which can be arranged in $4!$ ways and the three blue books can be arranged in $3!$ ways.

$$P(\text{blue and red together}) = \frac{2! \times 3! \times 4!}{7!} = \frac{2}{35}$$

9. The probability that Thandiswa will go dancing on a Saturday night (event D) is $0{,}6$ and the probability that she will go watch a movie is $0{,}3$ (event M). Determine the probability that she will:

   a) go dancing and watch a movie if D and M are independent.

      Solution:

$$\begin{aligned} P(D \text{ and } M) &= P(D) \times P(M) \\ &= 0{,}6 \times 0{,}3 = 0{,}18 \end{aligned}$$

   b) go dancing or watch a movie if D and M are mutually exclusive.

      Solution:

$$\begin{aligned} P(D \text{ or } M) &= P(D) + P(M) \\ &= 0{,}6 + 0{,}3 = 0{,}9 \end{aligned}$$

   c) go dancing and watch a movie if $P(D \text{ or } M) = 0{,}7$.

      Solution:

$$\begin{aligned} P(D \text{ and } M) &= P(D) + P(M) - P(D \text{ or } M) \\ &= 0{,}6 + 0{,}3 - 0{,}7 = 0{,}2 \end{aligned}$$

   d) not go dancing or go to a movie if $P(D \text{ and } M) = 0{,}8$.

      Solution:

$$\begin{aligned} P(\text{not } (D \text{ and } M)) &= 1 - P(D \text{ and } M) \\ &= 1 - 0{,}8 = 0{,}2 \end{aligned}$$

10. Three boys and four girls sit in a row.

    a) In how many ways can they sit in the row?

       Solution:

$$7! = 5040$$

    b) What is the probability that they sit in alternating gender positions?

       Solution:

       There is only one way they can sit alternately: GBGBGBG

       The number of ways they can sit alternately $= 1! \times 3! \times 4!$

$$P(\text{sit in alternating positions}) = \frac{1! \times 3! \times 4!}{7!} = \frac{1}{35}$$

11. The number plate on a car consists of any 3 letters of the alphabet (excluding the vowels, J and Q), followed by any 3 digits from 0 to 9. For a car chosen at random, what is the probability that the number plate starts with a Y and ends with an odd digit? Write your answer as a fraction.

    Solution:

    - The number plate starts with a Y, so there is only 1 choice for the first letter.

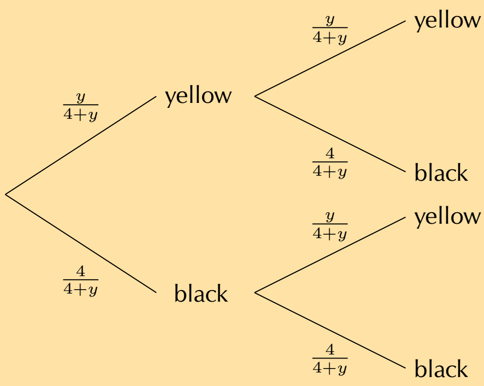
- The number plate ends with an odd digit, so there are 5 choices (1, 3, 5, 7, 9)
- There are 19 letters available because the 5 vowels (A, E, I, O, U), J and Q are excluded.

$$\begin{aligned} n(\text{plates starting with Y, ending with odd digit}) &= 1 \times 19^2 \times 10^2 \times 5 \\ &= 180\ 500 \\ n(\text{total possible number plates} &= 19^3 \times 10^3 \\ &= 6\ 859\ 000 \\ \therefore P(\text{plate starting with Y, ending with odd digit}) &= \frac{180\ 500}{6\ 859\ 000} \\ &= \frac{1}{38} \end{aligned}$$

12. There are four black balls and y yellow balls in a bag. Thandi takes out a ball, notes its colour and then puts it back in the bag. She then takes out another ball and also notes its colour. If the probability that both balls have the same colour is $\frac{5}{8}$, determine the value of y.

Solution:

Using a tree diagram, the different outcomes and probabilities can be illustrated as follows:



Solving for y:

$$\begin{aligned} \frac{5}{8} &= \frac{y}{4+y} \times \frac{y}{4+y} + \frac{4}{4+y} \times \frac{4}{4+y} \\ \frac{5}{8} &= \left(\frac{y}{4+y}\right)^2 + \left(\frac{4}{4+y}\right)^2 \\ \frac{5}{8} &= \frac{y^2+4^2}{(4+y)^2} \\ 5(4+y)^2 &= 8(y^2+16)) \\ 5(16+8y+y^2) &= 8y^2+128 \\ 80+40y+5y^2 &= 8y^2+128 \\ \text{Therefore } 3y^2-40y+48 &= 0 \\ (3y-4)(y-12) &= 0 \\ \text{Therefore } y &= 12(y \neq \frac{4}{3} \text{ as balls cannot be fractions}) \end{aligned}$$

13. A rare kidney disease affects only 1 in 1000 people and the test for this disease has a 99% accuracy rate.

   a) Draw a two-way contingency table showing the results if 100 000 of the general population are tested.

Solution:

If 100 000 people are tested and the prevalence rate is 0,1%:

$$\begin{aligned} 100\ 000 \times 0{,}001 &= 100 \text{ people are expected to be sick} \\ \text{Therefore } 100\ 000 - 100 &= 99\ 900 \text{ people are expected to be healthy} \end{aligned}$$

	Sick	**Healthy**	**Total**
Positive			
Negative			
Total	100	99 900	100 000

If the test is 99% accurate:

$$\begin{aligned} 100 \times 0{,}99 &= 99 \text{ sick people are expected to test positive} \\ \therefore 100 - 99 &= 1 \text{ sick person is expected to test negative} \\ \text{And } 99\ 900 \times 0{,}99 &= 98\ 901 \text{ healthy people are expected to test negative} \\ \therefore 99\ 900 - 98\ 901 &= 999 \text{ healthy people are expected to test positive} \end{aligned}$$

	Sick	**Healthy**	**Total**
Positive	99	999	1098
Negative	1	98 901	98 902
Total	100	99 900	100 000

b) Calculate the probability that a person who tests positive for this rare kidney disease is sick with the disease, correct to two decimal places.

Solution:

$$\begin{aligned} P(\text{sick if tested positive}) &= \frac{n(\text{sick and tested positive})}{n(\text{tested positive})} \\ &= \frac{99}{1098} \\ &= 0{,}09 \end{aligned}$$

Notice that this means that a positive result is wrong 91% of the time! This is an important concept in medical science. For very rare diseases, tests have to be highly accurate otherwise the result is meaningless.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29JP 2. 29JQ 3. 29JR 4. 29JS 5. 29JT 6. 29JV
7. 29JW 8. 29JX 9. 29JY 10. 29JZ 11. 29K2 12. 29K3
13. 29GT

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